DIFFERENTIAL & INTEGRAL CALCULUS.

WOLSTENHOLME.

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DIFFERENTIAL AND INTEGRAL CALCULUS.

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FIRST PRINCIPLES

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS,

AND THEIR APPLICATIONS, ACCORDING TO THE COURSE OF STUDY OF

COOPERS HILL COLLEGE.

BY

JOSEPH WOLSTENHOLME, M.A.,

PROFESSOR OF MATHEMATICS AT THE
INDIAN CIVIL ENGINEERING COLLEGE, COOPERS HILL;
LATE FELLOW AND TUTOR OF CHRIST'S COLLEGE, CAMBRIDGE:

TO WHICH IS ADDED

ELEMENTARY PROPOSITIONS IN THE THEORY OF COUPLES.

BY

ALFRED GEORGE GREENHILL, M.A.,

FELLOW AND MATHEMATICAL LECTURER OF EMMANUEL COLLEGE, CAMBRIDGE;
SOMETIME PROFESSOR OF APPLIED MATHEMATICS AT THE
I. C. E. COLLEGE, AFORESAID.

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INTRODUCTORY NOTE.

As there is no lack of works on the subject of the Differential and Integral Calculus to which no reasonable objection can be made, it may appear somewhat presumptuous in me to have added to the number. The usual text-books, however, all contain a great deal of matter which is superfluous when looked at from the point of view of those for whom this little book is especially intended, and I have put together the following pages with strict reference to the course of study at Coopers Hill College. I have given here and there a paper of questions which I have endeavoured to make of such a character that the student who finds he is able to answer them may be secure that he has satisfactory knowledge of the subject.

As the Theory of Couples is not sufficiently developed in our elementary text-books on Statics, the fundamental propositions in the theory are here given, as arranged by Mr. A. G. Greenhill when he was my colleague.

COOPERS HILL COLLEGE,
Sept. 30, 1874.

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DIFFERENTIAL AND INTEGRAL CALCULUS.

THE first thing necessary in beginning the study of the Differential Calculus, is to have a clear idea of what is meant by the *limit* of a varying quantity, which, under certain circumstances, assumes the unmeaning form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The definition of a limit in such a case is as follows: if U be a varying quantity, which on a certain hypothesis assumes an unmeaning form, and A the *limit* of this quantity, then as U is approaching its ultimate form, the difference between U and A continually diminishes, and may be made less than any assignable quantity however small, before U assumes the form $\frac{0}{0}$; but this difference does not become absolute zero before that point.

Of course we might say that, *e.g.*, when x approaches the value 2, x^2 approaches the value 4, and thus that 4 is the limit of x^2 when x approaches 2, but in such a case, where the function (x^2) has always an intelligible meaning, it is simpler to say that x^2 *is* 4 when $x=2$; but suppose we have $x=2+\frac{1}{z^2}$, and therefore $x^2=4+\frac{4}{z^2}+\frac{1}{z^4}$, then as z is indefinitely increased x continually approaches 2, and x^2 continually approaches 4, and the differences $x-2$, x^2-4 may be made less than any assignable quantity however small; and in such a case we should say that x has 2, and x^2 has 4, for its limit, since in this case x is not absolutely $=2$, nor $x^2=4$, for any finite value of z however large.

Good examples of limits are $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$, when x tends to 0, for we know that x being the circular measure of an angle $\frac{\sin x}{x} < 1$ and $> 1 - \frac{x^2}{4}$, whence $1 - \frac{\sin x}{x} > 0$ and $< \frac{x^2}{4}$, whence the difference $1 - \frac{\sin x}{x}$ continually diminishes as x diminishes, and may be made less than any assignable quantity, however small, before $x = 0$; whence the limit of $\frac{\sin x}{x}$ is 1. When x is absolute zero, $\frac{\sin x}{x}$ is unmeaning. This result is however, for shortness, written $\left(\frac{\sin x}{x}\right)_{x=0} = 1$; but the true meaning of such equations should always be borne in mind.

So also $\frac{\tan x}{x} = \frac{x}{\cos x}\,\frac{\sin x}{x}$; therefore, the limit of $\frac{\tan x}{x}$ = the limit of $\frac{\sin x}{x} = 1$, which, as before, is written $\left(\frac{\tan x}{x}\right)_{x=0} = 1$. These results are best illustrated by drawing the curves $y = \sin x$, $y = \tan x$; *i.e.* measure off on the straight line Ox any length OM (fig. 1) containing as many units of length as x contains of angle (in circular measure), and then draw MP at right angles to Ox and representing on the same scale $\sin x$, then as x increases from 0 to $\frac{1}{2}\pi$, y increases from 0 to 1 and is positive; from $x = \frac{1}{2}\pi$ to π, y decreases again from 1 to 0, and is positive, while from π to 2π, the same arithmetical values recur in the same order, but $\sin x$ is negative, and the corresponding curve is on the negative side of the axis of x. Here $\frac{\sin x}{x}$ is always represented by $\tan POM$, and since the limit has been shewn to be 1, the limiting value of the angle POM, when M moves up to A, is $\frac{1}{4}\pi$; but the limiting position of PO when M, and therefore P moves up to O, is what we mean by the tangent line to the curve at O. Hence

the curve crosses the axis of x at O at an angle of $\frac{1}{4}\pi$; and also does the curve $y = \tan x$ (fig. 2).

Another good illustration of the meaning of the word limit is given by considering what is meant by saying that a railway train, which may be continually varying its speed, is at any given moment moving at the rate of so many miles per hour. Every one, I believe, has a very clear conception that this is so—that at any one moment the train is going at one particular speed; but if we try to see how this is to be defined we are led at once to that particular kind of limit which is called a differential coefficient.

Suppose that during t minutes the train has gone over s yards, then if the rate were uniform $\frac{s}{t}$ would be the number of yards described in one minute, and this is true however large or however small t may be; whereas, if the speed be variable, $\frac{s}{t}$ will be continually changing, and will only represent the *average* velocity during the portion of time t. In this case if we take t continually smaller and smaller, this fraction will approximate more and more nearly to the velocity which the train has at the middle of the time t; and the speed at any particular moment will be the limit of this fraction when t is indefinitely diminished. Thus, to fix the ideas, suppose the train is so moving that the space described during any time t, measured from a certain epoch, shall vary as the square of t, say $s = at^2$, then if during an additional time t' the space described be s', we shall have $s + s'$ described in the whole time $t + t'$, or $s + s' = a(t + t')^2$, hence $s' = 2att' + at'^2$, or $\frac{s'}{t'} = 2at + at'$; *i.e.* the average velocity during the time t' is $2at + at'$, and diminishing t' indefinitely we obtain the rate at the end of the time t to be $2at$. If we now change our notation a little, putting Δs for s', Δt for t', we have $\frac{\Delta s}{\Delta t} = 2at + a.\Delta t$, and if we denote the fact of taking the

limit by putting d for Δ we obtain $\frac{ds}{dt}=2at$, and we have found the differential coefficient of the *function* at^2 of the *independent variable* t. If then $y=ax^2$, we shall have $\frac{dy}{dx}=2ax$.

The first notion of a differential coefficient was that of a velocity, time being in its nature an independent variable which we cannot conceive except as uniformly increasing, x the space described in a given time, then the speed, or velocity, at the end of that time was called the fluxion of x, and denoted by $\dot{x}$.

The general definition of a *differential coefficient* is as follows; if y be any quantity depending on another magnitude x in such a way that the number representing y can be expressed in terms of the number representing x, by such an equation as $y=\phi(x)$, then the limit of $\frac{\phi(x+h)-\phi(x)}{h}$ when h is indefinitely diminished, is the *differential coefficient*, or first *derived function*, of y with respect to x, and it is denoted by either of the symbols $\frac{dy}{dx}$, $\phi'(x)$.

This quantity will of course itself be a function of x, and will have different values when different values are given to x, thus $\phi'(a)$ denotes the value of $\phi'(x)$ when x is put $=a$, $\phi'(0)$ its value when x is put $=0$. If we repeat the operation, the result is called the second differential coefficient, or second *derived function* of x, and it is written either $\frac{d^2y}{dx^2}$, or $\phi''(x)$, and so on for any number of times.

There is no harm in writing the equation $\frac{dy}{dx}=\phi'(x)$ in the form $dy=\phi'(x)\,dx$, provided we bear in mind that this means that the two members of the equation $\Delta y=\phi'(x)\,\Delta x$ tend to have to each other a ratio of equality when both are diminished indefinitely. When

such an equation is used the members are called *differentials* of y, and of $\phi(x)$ respectively.

Another most useful way of considering a differential coefficient is by drawing the curve $y=\phi(x)$, which can always be done, or conceived to be done; for whatever the relation between x and y be, we can by measuring off a sufficient number of values of x along a fixed straight line, and all the corresponding values of y at right angles to them, obtain as many points as shall give a clear notion of the form of the curve.

Suppose then $OM=x$, $MP=\phi(x)$, $MN=h$ (fig. 3); therefore $ON=x+h$, $NQ=\phi(x+h)$, and draw PR perpendicular to NQ, then $RQ=\phi(x+h)-\phi(x)$, or if QP meet the axis of x in U, $\tan PUx=\dfrac{\phi(x+h)-\phi(x)}{h}$. Now when Q moves up to P, the limiting position of the straight line QPU is the tangent PT to the curve at P, or $\tan PTx = \text{limit}\ \dfrac{\phi(x+h)-\phi(x)}{h} = \dfrac{dy}{dx}$ or $\phi'(x)$.

Of course if we solved the equation for x and obtained it in the form $x=\psi(y)$, the curve represented by this equation would be the same as before, since all the values which satisfy one must satisfy the other, and since we should then have exactly as above $\tan PT'y=\dfrac{dx}{dy}$, we get $\dfrac{dy}{dx}\dfrac{dx}{dy}=1$.

We will now proceed to find the differential coefficients of simple functions of x.

(1) x^n, n being constant, *i.e.* not changing as x changes.

Taking $y=x^n$, we have

$$y+\Delta y=(x+\Delta x)^n \text{ or } (x+h)^n, \text{ or } \Delta y=(x+h)^n-x^n,$$

and
$$\frac{\Delta y}{\Delta x}=\frac{(x+h)^n-x^n}{h}=\frac{x^n}{h}\left\{\left(1+\frac{h}{x}\right)^n-1\right\}.$$

Now the expansion of $(1+z)^n$ by the Binomial Theorem is arithmetically true whenever z is numerically less than

1, whatever may be the value of n; hence since h has here to diminish indefinitely, $\frac{h}{x}$ must be numerically less than 1, before arriving at the ultimate hypothesis $h=0$, or

$$\left(1+\frac{h}{x}\right)^n - 1 = n\frac{h}{x} + \frac{n(n-1)}{\lfloor 2}\frac{h^2}{x^2} + \ldots;$$

therefore $\dfrac{\Delta y}{\Delta x} = nx^{n-1}\left\{1 + \dfrac{n-1}{2}\dfrac{h}{x} + \dfrac{(n-1)(n-2)}{\lfloor 3}\dfrac{h^2}{x^2} + \ldots\right\}$.

The quantity in brackets terminates when n is an integer, but has an infinite number of terms when n is fractional or negative, but is in all cases convergent when $\frac{h}{x} < 1$, and reduces to 1 when h and therefore $\frac{h}{x} = 0$, hence $\dfrac{dy}{dx} = nx^{n-1}$.

(2) $y = a^x$, $y + \Delta y = a^{x+h}$, $\Delta y = a^x(a^h - 1)$, $\dfrac{\Delta y}{\Delta x} = a^x . \dfrac{a^h - 1}{h}$,

or $$\frac{dy}{dx} = a^x \times \text{limit of } \left(\frac{a^h-1}{h}\right)_{h=0}.$$

Now $$a^h = 1 + h\log a + \frac{h^2}{\lfloor 2}(\log a)^2 + \ldots;$$

therefore $\dfrac{a^h - 1}{h} = \log a\left\{1 + \dfrac{h}{2}\log a + \dfrac{h^2}{2.3}(\log a)^2 + \ldots\right\}$.

But the series

$$1 + \frac{h}{2}\log a + \frac{h^2}{2.3}(\log a)^2 + \ldots,$$

is > 1 and $< 1 + \dfrac{h}{2}\log a + \left(\dfrac{h}{2}\log a\right)^2 + \left(\dfrac{h}{2}\log a\right)^3 + \ldots,$

and therefore $\qquad > 1$ and $< \dfrac{1}{1 - \dfrac{h}{2}\log a}$,

when h has been so far diminished that $\frac{h}{2}\log a < 1$, and

since these limits each $=1$ when $h=0$, we have limit $\left(\frac{a^h-1}{h}\right)_{h=0}=\log a$, and $\frac{dy}{dx}=a^x \log a$.

(3) $$y=\log_a x,\ y+\Delta y=\log_a(x+h),$$

$$\Delta y=\log_a(x+h)-\log_a x=\log_a\left(1+\frac{h}{x}\right),$$

$$\frac{\Delta y}{\Delta x}=\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)=\frac{1}{h}\left\{\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{3}\frac{h^3}{x^3}-\ldots\right\}\cdot\frac{1}{\log a}$$
$$=\frac{1}{x\log a}\left\{1-\frac{1}{2}\frac{h}{x}+\frac{1}{3}\frac{h^2}{x^2}-\ldots\right\},$$

whence $\frac{dy}{dx}=\frac{1}{x\log a}$, the series in brackets being easily proved as in the last to reduce to 1, when the limit is taken.

(2) and (3) can be obtained independently for such students as have not read the exponential series.

$$\frac{d}{dx}(a^x)\text{ has been shewn }=a^x\times\left(\frac{a^h-1}{h}\right)_{h=0},$$

$$\text{and }\frac{d}{dx}(\log_a x)=\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)_{h=0}.$$

We will find the latter limit first, as the former is immediately deducible from it.

Putting $h=xz$, we have

$$\frac{1}{xz}\log_a(1+z)\text{ or }\frac{1}{x}\log_a(1+z)^{\frac{1}{z}},$$

and since when $h=0$, $z=0$, we want to find the value of the limit $(1+z)^{\frac{1}{z}}_{z=0}$.

Now, since z is to be ultimately 0, we may assume it <1, and therefore expand $(1+z)^{\frac{1}{z}}$ by the Binomial Theorem, giving us

$$1+\frac{1}{z}z+\frac{\frac{1}{z}\left(\frac{1}{z}-1\right)}{1.2}z^2+\ldots,$$

or

$$1+1+\frac{1-z}{2}+\frac{(1-z)(1-2z)}{2.3}+\frac{(1-z)(1-2z)(1-3z)}{2.3.4}+\dots.$$

But the limits of $1-z$, $1-2z$, $1-3z\dots$, $1-nz$, are all 1 when $z=0$, for all finite values of n; hence the required limit is the limit of the sum of the series

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots \text{ to } \infty,$$

a certain abstract number which is usually denoted by ε.

Hence $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a\varepsilon$, or $\frac{1}{x\log a}$, the base ε being supposed when none is written.

Now to find $\left(\frac{a^h-1}{h}\right)_{h=0}$, take the numerator $=u$, and therefore $h=\log_a(1+u)$, and when $h=0$, $u=0$, hence $\left(\frac{a^h-1}{h}\right)_{h=0}=\left\{\frac{u}{\log_a(1+u)}\right\}_{u=0}=\frac{1}{\log_a\varepsilon}$ from the limit just previously found; whence $\frac{d}{dx}(a^x)=\frac{a^x}{\log_a\varepsilon}=a^x\log a$.

(4) $y=\sin mx$; $\Delta y=\sin m(x+h)-\sin mx$

$$=2\sin\frac{mh}{2}\cos\left(mx+\frac{mh}{2}\right);$$

$$\frac{\Delta y}{\Delta x}=\frac{2\sin\frac{mh}{2}}{h}\cos\left(mx+\frac{mh}{2}\right)=m\left(\frac{\sin\frac{mh}{2}}{\frac{mh}{2}}\right)\cos\left(mx+\frac{mh}{2}\right),$$

or $$\frac{dy}{dx}=m\cos mx, \text{ the limit of } \frac{\sin\frac{mh}{2}}{\frac{mh}{2}} \text{ being } 1.$$

(5) $y=\cos mx$,

$$\Delta y=\cos m(x+h)-\cos mx, =-2\sin\frac{mh}{2}\sin\left(mx+\frac{mh}{2}\right),$$

$$\frac{\Delta y}{\Delta x} = -m \frac{\sin\frac{mh}{2}}{\frac{mh}{2}} \sin\left(mx + \frac{mh}{2}\right);$$

therefore $$\frac{dy}{dx} = -m \sin mx.$$

(6) $$y = \tan mx,$$

$$\Delta y = \tan m(x+h) - \tan mx = \frac{\sin mh}{\cos mx \cos m(x+h)},$$

$$\frac{\Delta y}{\Delta x} = m \frac{\sin mh}{mh} \cdot \frac{1}{\cos mx \cos(mx + mh)};$$

therefore $$\frac{dy}{dx} = \frac{m}{\cos^2 mx} = m \sec^2 mx \quad \text{or} \quad = m(1 + \tan^2 mx).$$

(7) $$y = \cot mx,$$

$$\Delta y = \cot m(x+h) - \cot mx = -\frac{\sin mh}{\sin m(x+h) \sin mx},$$

$$\frac{\Delta y}{\Delta x} = -m \frac{\sin mh}{mh} \cdot \frac{1}{\sin mx \sin(mx + mh)};$$

or $$\frac{dy}{dx} = \frac{-m}{\sin^2 mx} \quad \text{or} \quad -m(1 + \cot^2 mx).$$

(8) $$y = \sec mx,$$

$$\Delta y = \sec m(x+h) - \sec mx$$

$$= \frac{1}{\cos m(x+h)} - \frac{1}{\cos mx} = \frac{2 \sin\frac{mh}{2} \sin\left(mx + \frac{mh}{2}\right)}{\cos mx \cos m(x+h)},$$

$$\frac{\Delta y}{\Delta x} = m \cdot \frac{\sin\frac{mh}{2}}{\frac{mh}{2}} \frac{\sin\left(mx + \frac{mh}{2}\right)}{\cos mx \cos m(x+h)},$$

or $$\frac{dy}{dx} = \frac{m \sin mx}{\cos^2 mx} = m \sec mx \tan mx.$$

The inverse trigonometrical functions may be differentiated by means of the results found for the direct ones; but to avoid the ambiguity ± it is advisable to lay down

the rule that $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, shall always be interpreted as acute angles, positive or negative; *i.e.* as lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. This rule will avoid many perplexities and risk of errors.

(9) $$y = \sin^{-1}\left(\frac{x}{a}\right);$$

therefore $x = a \sin y$, $x + \Delta x = a \sin(y + \Delta y)$,

or $\Delta x = a\{\sin(y+\Delta y) - \sin y\} = 2a \sin\frac{\Delta y}{2}\cos\left(y + \frac{\Delta y}{2}\right)$.

$$1 = a\frac{\Delta y}{\Delta x}\cdot\frac{\sin\frac{\Delta y}{2}}{\frac{\Delta y}{2}}\cos\left(y+\frac{\Delta y}{2}\right);$$

therefore $1 = a\dfrac{dy}{dx}\cos y$, $\dfrac{dy}{dx} = \dfrac{1}{a\cos y} = \dfrac{1}{\sqrt{(a^2 - x^2)}}$.

The angle y being by our rule between $+\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, its cosine is positive. If that rule be not followed we must write

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{(a^2 - x^2)}}.$$

(10) $y = \cos^{-1}\dfrac{x}{a}$, $x = a\cos y$, whence as in the last

$$1 = -a\frac{dy}{dx}\sin y, \quad \frac{dy}{dx} = \frac{-1}{\sqrt{(a^2 - x^2)}}.$$

This might also have been deduced from the identity

$$\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right) = \tfrac{1}{2}\pi.$$

(11) $$y = \tan^{-1}\left(\frac{x}{a}\right), \quad x = a\tan y,$$

$$1 = a\frac{dy}{dx}\sec^2 y, \quad \frac{dy}{dx} = \frac{a}{a^2 + x^2}.$$

(12) $$y = \cot^{-1}\left(\frac{x}{a}\right) = \tfrac{1}{2}\pi - \tan^{-1}\frac{x}{a};$$

therefore $$\frac{dy}{dx} = -\frac{a}{a^2+x^2}.$$

(13) $$y = \sec^{-1}\left(\frac{x}{a}\right), \quad x = a \sec y,$$

$$1 = a\frac{dy}{dx}\sec y \tan y = \frac{dy}{dx}\, x \sqrt{\left(\frac{x^2}{a}\right) - 1},$$

or $$\frac{dy}{dx} = \frac{a}{x\sqrt{(x^2-a^2)}}.$$

Students often find it difficult to remember when and where to put the a which occurs in some of these differential coefficients. This can always be settled by considering x and a to represent the lengths of certain lines, which in fact they generally do. In all these cases (9)...(13), y being the circular measure of an angle is a pure abstract number, whence $\frac{\Delta y}{\Delta x}$, and therefore $\frac{dy}{dx}$ must be a symbol represented by an abstract number $\div$ a number representing the length of some line (where the unit of length may be any whatever), or in other words each result is of -1 dimensions in space. Hence, when $\sqrt{(a^2-x^2)}$ is the denominator, the numerator will be a number only, or no a is wanted; but when the denominator is a^2+x^2, or $x\sqrt{(x^2-a^2)}$, we must have a symbol in the numerator which is of one dimension in space, and the a is required. The proper sign to be affixed can be always supplied by considering whether the function is one which increases or diminishes as x increases. If the former, the sign is of course positive, since Δy, Δx must be then of the same sign, and therefore $\frac{dy}{dx}$ is positive, if otherwise negative. Thus

$$\frac{d}{dx}\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}, \frac{d}{dx}\left\{\tan^{-1}\left(\frac{x}{a}\right)\right\} \text{ are positive,}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(\frac{x}{a}\right)\right\}, \frac{d}{dx}\left\{\cot^{-1}\left(\frac{x}{a}\right)\right\} \text{ negative.}$$

Differentiation of Complex Functions.

The differentiation of complex functions can always be made to depend upon that of simple functions by the following method.

Let y be a function of z, z being a function of x, and suppose that when x receives an increment Δx, z receives an increment Δz, and y in consequence an increment Δy; then since $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta z}\dfrac{\Delta z}{\Delta x}$ always, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$. So also $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dz}\dfrac{dx}{dz}$, and so on through any number of steps that may be necessary before arriving at the simple function of x. These processes should, however, be performed by a mental operation only; thus, if $y = \log \sin x$, we may put $\sin x = z$, and therefore $y = \log z$, whence

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{z}\cos x = \frac{\cos x}{\sin x} = \cot x.$$

This should be *written*

$$\frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \cot x.$$

So $\dfrac{d}{dx}\log\{\tan(\frac{1}{4}\pi + \frac{1}{2}x)\} = \dfrac{1}{\tan(\frac{1}{4}\pi + \frac{1}{2}x)}\dfrac{d}{dx}\{\tan(\frac{1}{4}\pi + \frac{1}{2}x)\}$

$$= \cot(\tfrac{1}{4}\pi + \tfrac{1}{2}x)\sec^2(\tfrac{1}{4}\pi + \tfrac{1}{2}x)\frac{d}{dx}(\tfrac{1}{4}\pi + \tfrac{1}{2}x)$$

$$= \cot(\tfrac{1}{4}\pi + \tfrac{1}{2}x)\sec^2(\tfrac{1}{4}\pi + \tfrac{1}{2}x)\,.\,\tfrac{1}{2} = \frac{1}{2\sin(\frac{1}{4}\pi + \frac{1}{2}x)\cos(\frac{1}{4}\pi + \frac{1}{2}x)}$$

$$= \frac{1}{\sin 2(\frac{1}{4}\pi + \frac{1}{2}x)} = \frac{1}{\sin(\frac{1}{2}\pi + x)} = \frac{1}{\cos x}.$$

The differential coefficient of the sum of a number of different functions of x is obviously the sum of their differential coefficients; that of a constant is zero, and that of a function of x multiplied by a constant is the differential

coefficient of the function, multiplied by the same constant.

(For if $y = u_1 + u_2 + u_3 + \dots$, $\Delta y = \Delta u_1 + \Delta u_2 + \dots$,

if $$y = a,\ \Delta y = 0;$$

and if $$y = au,\ \Delta y = a\Delta u,$$

and the values of $\frac{dy}{dx}$ are $\frac{du_1}{dx} + \frac{du_2}{dx} + \dots$, zero, and $a\,\frac{du}{dx}$ respectively). The differential coefficient of the product of any number of different functions of x is formed by differentiating each function as if all the others were constant, and adding the results; for if

$$y = u_1 u_2 \dots u_n,\ \log y = \log u_1 + \log u_2 + \dots + \log u_n,$$

therefore $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \dots + \frac{1}{u_n}\frac{du_n}{dx},$$

or $$\frac{dy}{dx} = \frac{du_1}{dx} u_2 u_3 \dots u_n + u_1 \frac{du_2}{dx} u_3 \dots u_n + \dots + u_1 u_2 \dots u_{n-1} \frac{du_n}{dx}.$$

The rule for a quotient may be deduced from this, or may be found by the same process, thus if

$$y = \frac{u}{v},\ \log y = \log u - \log v,\ \text{ or } \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{1}{v}\frac{dv}{dx};$$

therefore $$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2},$$

the result, which is often put into words as a rule to be remembered, but, in my opinion, that process by no means makes the rule easier to apply, and it is very doubtful if it is worth while having a separate rule for a quotient. Certainly if the index of the denominator be other than unity, it is rather better to differentiate the whole function considering it a product. Thus if $y = \frac{x^{\frac{3}{2}}}{(a+x)^{\frac{5}{2}}}$, we get, by the rule for quotients, $\frac{dy}{dx} = \frac{(a+x)^{\frac{5}{2}}\ \frac{3}{2}x^{\frac{1}{2}} - x^{\frac{3}{2}}\ \frac{5}{3}(a+x)^{\frac{3}{2}}}{(a+x)^5}$,

from which the useless factor $(a+x)^{\frac{3}{2}}$ must be divided out; now, differentiating it as $x^{\frac{3}{2}}(a+x)^{-\frac{5}{2}}$, we have

$$\tfrac{3}{2}x^{\frac{1}{2}}(a+x)^{-\frac{5}{2}} - \tfrac{5}{2}x^{\frac{3}{2}}(a+x)^{-\frac{7}{2}} = \frac{3x^{\frac{1}{2}}(a+x) - 5x^{\frac{3}{2}}}{2(a+x)^{\frac{7}{2}}} = \frac{x^{\frac{1}{2}}(3a-2x)}{2(a+x)^{\frac{7}{2}}}.$$

Probably, however, it is better still to take the logarithm of both sides before differentiating, as is done in proving the rule, thus

$$\log y = \tfrac{3}{2}\log x - \tfrac{5}{2}\log(a+x);$$

therefore $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\frac{3}{x} - \frac{5}{a+x}\right\} = \frac{3a-2x}{2x(a+x)},$$

and $$\frac{dy}{dx} = \frac{(3a-2x)\,x^{\frac{1}{2}}}{2(a+x)^{\frac{7}{2}}}.$$

The last method certainly gives the least chance for mistake.

An expression of the form u^v, where u and v are both functions of x, is best differentiated by taking its logarithm. If

$$y = u^v, \quad \log y = v\log u;$$

therefore $$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}\log u + \frac{v}{u}\frac{du}{dx},$$

or $$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log u \frac{dv}{dx}.$$

It is to be noticed that the two terms of this expression are what we should have obtained by differentiating the whole expression considering (1) v constant, (2) u constant, and adding the results together; for if u *were* constant, the differential coefficient of u^v *would be* $u^v \log u \dfrac{dv}{dx}$, and if v *were* constant, $vu^{v-1}\dfrac{du}{dx}$. This is indeed an invariable rule; if y be a complex function of x, then, in whatever way the various simple functions, of which it is composed, be connected together, the complete differential coefficient of y is the sum of all of what we may call *partial* diffe-

rential coefficients obtained severally by considering all the functions but one to be constant. All the above rules are included in this. This may be expressed as follows: if $y = F(u, v, w, \ldots)$, where u, v, w are functions of x, $\frac{dy}{dx} = \left(\frac{dF}{du}\right)\frac{du}{dx} + \left(\frac{dF}{dv}\right)\frac{dv}{dx} + \ldots$, the brackets signifying *partial* differentiation, viz. that in forming $\left(\frac{dF}{du}\right)$ all the functions $v, w, \ldots$ except u are regarded as constant. Thus, to differentiate x^{x^x}, (1) considering the upper x's to be constant, we get $x^x \times x^{x^x-1}$, (2) considering the centre x as the only variable, and the differential coefficient is $x^{x^x} \log x . x . x^{x-1}$ or $x^x x^{x^x} \log x$, (3) considering the upper x as the only variable, and we have $x^{x^x} \log x\,(x^x \log x)$ or $x^x x^{x^x} (\log x)^2$, whence the true value of $\frac{dy}{dx}$ is

$$x^x x^{x^x}\left\{\frac{1}{x} + \log x + (\log x)^2\right\}.$$

This may be tested by taking the logarithmic differential coefficient of both members of the equation $y = x^{x^x}$.

QUESTIONS ON THE PRECEDING.

1. Define the terms *function*, *independent variable*, and explain what is meant by the *limit* of a function which for a particular value of the variable assumes an unmeaning form. Prove that the limit of $\sec x - \tan x$, as x approaches the value $\frac{1}{2}\pi$, is 0; and that of $\sec^2 x \sec x \tan x$ is $\frac{1}{2}$.

2. Give examples of functions which are limited in either sign or magnitude although the independent variable on which they depend is capable of all values from $-\infty$ to ∞. *State* how any of the following functions are limited, x being capable of all values:

$$\sin x,\ \tan x,\ x^2 + \frac{1}{x^2},\ (1+x)^{\frac{1}{x}},\ \left(1+\frac{1}{x}\right)^x,\ \frac{x^2+x+1}{x^2+3x+4}.$$

3. Prove that the limit of $\left(1+\frac{1}{x}\right)^x$ when x is increased indefinitely is the limit of the sum of the series

$$1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}+\ldots \text{ to } \infty \text{ (or } \varepsilon\text{)};$$

and that the corresponding limit of $\left(1+\frac{a}{x}\right)^x$ is that of the sum of the series $1+a+\frac{a^2}{\underline{|2}}+\ldots+\frac{a^n}{\underline{|n}}+\ldots$ to ∞, and is also ε^a.

4. Deduce the limits of $\frac{1}{z}\log_a(1+z)$ and of $\frac{a^z-1}{z}$, when z is indefinitely diminished.

Prove that the limit of the product

$$\frac{2}{a^{\frac{1}{2}}+1}\,\frac{2}{a^{\frac{1}{4}}+1}\,\frac{2}{a^{\frac{1}{8}}+1}\cdots\frac{2}{a^{\frac{1}{2^n}}+1},$$

when n is indefinitely increased, is $\frac{\log a}{a-1}$.

5. Define the differential coefficient of a function of any independent variable. Give a geometrical illustration of the meaning of a differential coefficient, and prove that if $\phi(x)$ and $\phi'(x)$ be finite and continuous,

$$\phi(x+h)=\phi(x)+h\phi'(x+\theta h),$$

where θ is some proper fraction. Obtain the differential coefficients of $\sqrt{(x)}$, $(a+x)$, and $\sqrt{(a^2-x^2)}$.

6. Obtain the differential coefficients of

$$x^n,\ a^x,\ \varepsilon^x,\ \varepsilon^{a+bx},\ \log_a x, \text{ and } \log\{x+\sqrt{(1+x^2)}\}.$$

7. Differentiate

$$\sin x,\ \cos x,\ \tan x,\ \sin^3 x,\ \log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x),\ \log\tan\tfrac{1}{2}x,$$
$$\sin^{-1}(\sin x-\cos x), \text{ and } \log\{\sin x+\cos x+\sqrt{(\sin 2x)}\}.$$

8. Prove the rule for differentiating any product, or quotient, of different functions of x. Differentiate the product $(1+x+x^2)(1-x+x^2)(1-x^2+x^4)$, simplifying the result.

INTEGRAL CALCULUS. FIRST PRINCIPLES.

Having investigated methods of finding the differential coefficients of any functions of x, the next thing in the natural order of the subject (just as division succeeds multiplication) is to reverse the process, and to enquire after methods which shall enable us, when the differential coefficient is given, to determine the quantity of which it is the differential coefficient. This is a very different matter, and we cannot assert that it is always possible. If a function can be expressed in terms of x by any known algebraical symbols we can find its differential coefficient, but we can only reverse all these processes to furnish us with *integrals* of certain functions of x, and if we cannot by any of our methods reduce a quantity whose integral is sought to coincidence with a known result of differentiation, we are at a stand.

Thus having investigated the properties of the function of x which we call its logarithm, and having found that $\frac{d}{dx}(\log x)$ is $\frac{1}{x}$, we say (meaning the same thing, neither more nor less) that the integral of $\frac{1}{x}$ is $\log x$; but if we had not known the properties of this function we could not have integrated $\frac{1}{x}$. No one can integrate $\frac{1}{\sqrt{(1-x^4)}}$ and many other apparently simple functions.

The notation adopted in the Integral Calculus is as follows: if $\frac{dy}{dx} = \phi(x)$, and $y = \psi(x)$, then $\int \phi(x)\, dx = \psi(x)$, or more correctly $= \psi(x) + C$, where C may be any *constant* quantity; that is, any quantity independent of x. This constant is necessary since $\frac{d}{dx}\{\psi(x)\} = \frac{d}{dx}\{\psi(x) + C\}$. The reason of the notation will be explained subsequently; at present it will be better simply to accept the equation

$\int\phi(x)\,dx = \psi(x) + C$, as meaning that the differential coefficient $\psi'(x)$ of $\psi(x)$ is $\phi(x)$; or, in other words, that the integral of $\phi(x)$ is $\psi(x) + C$. Such result is called an *indefinite integral*. If no other data are given, C cannot be determined, but every value will answer the question. Thus if $\frac{dy}{dx} = \frac{1}{2\sqrt{(x)}}$, we know that $\frac{d}{dx}\{\sqrt{(x)}\} = \frac{1}{2\sqrt{(x)}}$; and therefore $\frac{dy}{dx} = 2\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\{2\sqrt{(x)}\}$, and therefore $\frac{d}{dx}(y - 2\sqrt{x}) = 0$; whence we must have $y - 2\sqrt{(x)} = C$, where C may be any constant quantity. Suppose, however, we have the farther datum that when $x = 1$, $y = 0$, then we have $-2 = C$, *i.e.* $C = -2$, and the general relation is $y = -2 + 2\sqrt{(x)}$. Such a result is called a *corrected integral*. If, finally, the thing sought be the value of y when $x = 4$, then we get from this equation $y = 2$. This result is written

$$\int_1^4 \frac{dx}{\sqrt{(x)}} = 2 \text{ and is called a \textit{definite integral.}}$$

In general

$$\int\phi(x)\,dx = \psi(x) + C \text{ the \textit{indefinite integral.}}$$

$$\int_a^x \phi(x)\,dx = \psi(x) - \psi(a) \text{ \textit{corrected} integral.}$$

$$\int_a^b \phi(x)\,dx = \psi(b) - \psi(a) \text{ \textit{definite} integral.}$$

The case in which $C = 0$ is very frequent, and such a result is generally written $\int\phi(x)\,dx = \psi(x)\ (+C = 0)$. The best illustration of integration is given by considering the result sought for as the area of a curve bounded by the axis of x, any two ordinates, and the curve. Thus suppose $aPQb$ (fig. 4) to be a curve, the ordinate MP at any point being a known function $\phi(x)$ of OM or x the abscissa, take $ON = x + \Delta x$, $NQ = y + \Delta y = \phi(x + \Delta x)$,

then if U denote the area $aAMP$ measured from *any* fixed ordinate aA, then $U + \Delta U =$ area $aANQ$, or $\Delta U =$ area $PMNQ$, which for all forms of the curve, provided $\phi(x)$ is not infinite, will lie between the areas of the two rectangles PN, MQ or between $\phi(x) . \Delta x$ and $\phi(x + \Delta x) . \Delta x$; hence $\frac{\Delta U}{\Delta x}$ lies between $\phi(x)$ and $\phi(x + \Delta x)$; or

$$\frac{dU}{dx} = \phi(x) = y.$$

Now suppose $\psi(x)$ to be a function of x, such that $\psi'(x) = \phi(x)$, then

$$\frac{dU}{dx} = \frac{d}{dx}\{\psi(x)\}, \text{ or } U - \psi(x) = C,$$

i.e. $\qquad U = \psi(x) + C$ the *indefinite integral.*

The original differential equation is obviously true whatever be the position of aA, and hence the necessity of an undetermined constant in the integral. As soon as the initial ordinate Aa is chosen this constant is determined; for if $OA = a$, then when $x = a$, $U = 0$, or $0 = \psi(a) + C$, or $U = \psi(x) - \psi(a)$ the *corrected integral.* If the final ordinate be Bb, where $x = b$, we shall have $\psi(b) - \psi(a)$ the *definite integral*, which is written, before determination, as $\int_a^b \phi(x)\,dx$.

The meaning of the symbols $\int \phi(x)\,dx$ is the limit of the sum of an infinite number of magnitudes such as $\phi(x) . \Delta x$ when Δx is made indefinitely small, *i.e.* in the figure of such quantities as the rectangle PN, for if the whole base AB be divided into n equal parts, each $= \Delta x$, and on each be completed rectangles as PN, MQ, the sum of all the internal rectangles as PN will be less than the area of the curve $aABb$, and the sum of all the external rectangles as MQ will be greater; but the sum of the former or $\Sigma\{\phi(x)\,\Delta x\}$ is

$$\Delta x\,[\phi(a) + \phi(a + \Delta x) + \phi(a + 2\Delta x) + \ldots + \phi\{a + (n-1)\,\Delta x\}],$$

and of the latter or $\Sigma \{\phi(x+\Delta x) . \Delta x\}$ is

$$\Delta x \{\phi(a+\Delta x)+\phi(a+2\Delta x)+\ldots+\phi(a+n\Delta x)\},$$
$$a+n . \Delta x \text{ being } = b,$$

and the difference of these sums is $\Delta x \{\phi(a+n\Delta x)-\phi(a)\}$ or $\Delta x \{\phi(b)-\phi(a)\}$, which vanishes when $\Delta x = 0$; hence the area of the curve lying between these, which in the limit coincide, will be equal to the limit of either, or

$$\psi(b)-\psi(a) = \text{the limit of } \Delta x [\phi(a)+\phi(a+\Delta x)$$
$$+\phi(a+2\Delta x)+\ldots+\phi\{a+(n-1)\Delta x\}] \text{ when } \Delta x = 0,$$

and each element of which this sum is composed is of the form $\phi(x)\Delta x$, x having successively all values between a and b when Δx is made indefinitely small; the whole of which facts are succinctly and conveniently recorded by the notation $\int_a^b \phi(x) . dx$, the $\int$ being originally only the long s, and being used as being the initial letter of the word *sum*.

Every differential coefficient, which has been investigated, gives rise to a corresponding integral:

(1) $\frac{d}{dx}(x^m) = mx^{m-1}$; or $\int mx^{m-1}dx = x^m$, or $\int x^{m-1}dx = \frac{x^m}{m}$; but to apply this in integration it is more convenient to put $m+1$ for m, so that $\int x^m dx = \frac{x^{m+1}}{m+1} + C$.

(In simply transforming differential coefficients we shall omit the C, but it must on no account be omitted in any application).

(2) $\frac{d}{dx}(\log x) = \frac{1}{x}$; or $\int \frac{dx}{x} = \log x$.

Here it may be asked why this integral is not deduced at once as a particular case of (1) by putting $m = -1$, since then x^m becomes $\frac{1}{x}$. It will be seen that in that case $\frac{x^{m+1}}{m+1}$ becomes ∞ for all finite values of x, whence

we conclude that C must for that particular value of m have an infinite value, and we deduce the special case as follows: $\int x^m dx = \frac{x^{m+1}}{m+1} + C = \frac{x^{m+1}-1}{m+1} + C'$, since $\frac{1}{m+1}$ is also independent of x. Putting in this $m = -1$, we have

$$\int \frac{dx}{x} = C' + \text{limit of } \left(\frac{x^{m+1}-1}{m+1}\right)_{m+1=0}$$

$$= C' + \text{limit of } \left(\frac{x^u - 1}{u}\right)_{u=0} = C' + \log x.$$

(See the first chapter of *Differential Calculus*).

(3) $\frac{d}{dx}(a^x) = a^x \log a$;

therefore $a^x = \log a \int a^x dx$, or $\int a^x dx = \frac{a^x}{\log a}$.

These lead also to the integrals

$$\int (x+a)^m dx = \frac{(x+a)^{m+1}}{m+1}, \quad \int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b).$$

Of course, as in differentiation, a constant multiplier appears in the integral as a constant multiplier also, *i.e.*

$$\int a\phi(x)\,dx = a\int \phi(x)\,dx.$$

(4) $\int \frac{dx}{a^2 - x^2} = \int \frac{1}{2a}\left\{\frac{1}{a-x} + \frac{1}{a+x}\right\} dx$

$$= \frac{1}{2a}\{\log(a+x) - \log(a-x)\} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right).$$

This is if $x < a$; if $x > a$, we should have $\frac{1}{2a}\log\left(\frac{x+a}{x-a}\right)$. (These two forms differ by the constant, although unreal, quantity $\frac{1}{2a}\log(-1)$).

(5) $\frac{d}{dx}(\sin ax) = a \cos ax$, $\int \cos ax\,dx = \frac{1}{a}\sin ax$.

(6) $\frac{d}{dx}(\cos ax) = -a \sin ax$, $\int \sin ax\,dx = -\frac{1}{a}\cos ax$.

(7) $\dfrac{d}{dx}(\tan ax) = a \sec^2 ax, \quad \int \sec^2 ax\, dx = \dfrac{1}{a} \tan ax.$

(8) $$\frac{d}{dx}\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\} = \frac{1}{\sqrt{(a^2 - x^2)}},$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(\frac{x}{a}\right)\right\} = -\frac{1}{\sqrt{(a^2 - x^2)}};$$

therefore $\displaystyle\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\left(\frac{x}{a}\right)$, or $-\cos^{-1}\left(\dfrac{x}{a}\right)$;

(differing by $\frac{1}{2}\pi$).

(9) $$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{x}{a}\right)\right\} = \frac{a}{a^2 + x^2}, \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

(10) $$\frac{d}{dx}\log\left\{\frac{x + \sqrt{(x^2 \pm a^2)}}{a}\right\}$$

$$= \frac{1}{x + \sqrt{(x^2 \pm a^2)}} \cdot \left\{1 + \frac{x}{\sqrt{(x^2 \pm a^2)}}\right\} = \frac{1}{\sqrt{(x^2 \pm a^2)}};$$

therefore $$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log\left\{\frac{x + \sqrt{(x^2 \pm a^2)}}{a}\right\}.$$

(There is no absolute necessity for the a in the denominator of the log. It will be found in practice to be simpler to use this form than the one without the a).

These are the fundamental integrals and should be remembered; other integrals can, by various transformations, be made to depend upon some of these; but every integration is finally an act of the memory, and the different processes of integration only consist in bringing the integral operated upon into a form which the memory recognises. Practice only can render any one expert at the transformations required; but the general formula is as follows: Suppose $\dfrac{dy}{dx} = \phi(x)$, *i.e.* $y = \int \phi(x)\, dx$, then if we can simplify $\phi(x)$ by putting $x = f(z)$, we shall have $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$; therefore $\dfrac{dy}{dz} = \dfrac{dy}{dx}\dfrac{dx}{dz} = \phi\{f(z)\} \cdot f'(z)$; and therefore

$y = \int \phi \{f(z)\} . f'(z)\,dz$; *i.e.* we must substitute for x throughout the whole of the expression under the sign $\int$, putting $f'(z)\,dz$ for dx as well as $f(z)$ for x. As a good example take $\int \frac{dx}{\varepsilon^x + \varepsilon^{-x}}$, putting $\varepsilon^x = z$, and therefore $x = \log z$, $\frac{dx}{dz} = \frac{1}{z}$; therefore

$$\int \frac{dx}{\varepsilon^x + \varepsilon^{-x}} = \int \frac{\frac{dz}{z}}{z + \frac{1}{z}} = \int \frac{dz}{1 + z^2} = \tan^{-1}(z) = \tan^{-1}(\varepsilon^x).$$

This process may also be written, without using a new symbol, as follows:

$$\int \frac{dx}{\varepsilon^x + \varepsilon^{-x}} = \int \frac{\varepsilon^x dx}{\varepsilon^{2x} + 1} = \int \frac{d(\varepsilon^x)}{(\varepsilon^x)^2 + 1} = \tan^{-1}(\varepsilon^x).$$

The expressions $\frac{1}{\sqrt{(a + bx + cx^2)}}$, $\frac{1}{a + bx + cx^2}$, can be reduced to known forms by "completing the square," as if arranging the quadratic involved for solution; viz.

$$a + bx + cx^2 = c\left(x^2 + \frac{bx}{c} + \frac{b^2}{4c^2}\right) + a - \frac{b^2}{4c}$$

$$= c\left(x + \frac{b}{2c}\right)^2 + \frac{4ac - b^2}{4c}.$$

Hence, putting $x + \frac{b}{2c} = z$,

$$\int \frac{dx}{\sqrt{(a + bx + cx^2)}} = \int \frac{dz}{\sqrt{\left(cz^2 + \frac{4ac - b^2}{c}\right)}},$$

which is of the forms (8) or (10) according as c is positive or negative, and

$$\int \frac{dx}{a + bx + cx^2} = \int \frac{dx}{\frac{4ac - b^2}{c} + cz^2},$$

which is of the forms (4) or (9) according as c is positive or negative.

The integral $\int \frac{dx}{(x+a)\sqrt{(p+qx+rx^2)}}$ is reduced to one of the preceding by putting $x+a=\frac{1}{z}$.

The forms $\frac{px+q}{\sqrt{(a+bx+cx^2)}}$, and $\frac{px+q}{a+bx+cx^2}$, are also to be integrated by "completing the square," when they take the forms $\frac{p'z+q'}{\sqrt{(m+cz^2)}}$, $\frac{p'z+q'}{m+cz^2}$; and $\frac{z}{\sqrt{(m+cz^2)}}$, $\frac{z}{m+cz^2}$, become known forms by the substituting $z^2=u$. It is not at all worth while to complete the integration of all the special cases; the *method* by which they are reduced to known forms is what should be attended to. I will take an example of each form.

$$(\alpha) \quad \int \frac{x+4\,dx}{\sqrt{(x^2+4x+3)}} = \int \frac{(x+2)+2}{\sqrt{\{(x+2)^2-1\}}}\,dx$$

$$= \int \frac{(x+2)\,dx}{\sqrt{\{(x+2)^2-1\}}} + 2\int \frac{dx}{\sqrt{\{(x+2)^2-1\}}},$$

and

$$\int \frac{(x+2)\,dx}{\sqrt{(\overline{x+2}|^2-1)}} = \tfrac{1}{2}\int \frac{d\,\{(x+2)^2-1\}}{\sqrt{\{(x+2)^2-1\}}} = \sqrt{\{(x+2)^2-1\}},\ (1);$$

also
$$\int \frac{dx}{\sqrt{\{(x+2)^2-1\}}} = \int \frac{d\,(x+2)}{\sqrt{\{(x+2)^2-1\}}}$$

$$= \log[x+2+\sqrt{\{(x+2)^2-1\}}],\ (10);$$

therefore

$$\int \frac{x+4}{\sqrt{(x^2+4x+3)}}\,dx = \sqrt{(x^2+4x+3)} + 2\log\{x+2+\sqrt{(x^2+4x+3)}\}.$$

$$(\beta) \quad \int \frac{(x+4)\,dx}{x^2+4x+3} = \int \frac{(z+2)\,dz}{z^2-1}$$

$$= \int \frac{\frac{1}{2}d\,(z^2)}{z^2-1} + 2\int \frac{dz}{z^2-1}$$

$$= \tfrac{1}{2}\log(z^2-1) + \log\left(\frac{z-1}{z+1}\right)$$

$$= \log\frac{(z-1)^{\frac{3}{2}}}{(z+1)^{\frac{1}{2}}} = \tfrac{3}{2}\log(x+1) - \tfrac{1}{2}\log(x+3).$$

(Such however had better be done at once by assuming $\frac{x+4}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$.)

$$(\alpha')\ \int \frac{2x+3}{\sqrt{(3-2x-x^2)}}\,dx = \int \frac{2z+1}{\sqrt{(4-z^2)}}\,dz = \int \frac{d\,(z^2)}{\sqrt{(4-z^2)}} + \int \frac{dz}{\sqrt{(4-z^2)}}$$

$$= -2\sqrt{(4-z^2)} + \sin^{-1}\left(\frac{z}{2}\right) = -2\sqrt{(3-2x-x^2)} + \sin^{-1}\left(\frac{x+1}{2}\right).$$

$$(\beta')\ \int \frac{(2x+3)}{5+2x+x^2}\,dx = \int \frac{2z+1}{z^2+4}\,dz = \int \frac{d\,(z^2)}{z^2+4} + \int \frac{dz}{z^2+4}$$

$$= \log(z^2+4) + \tfrac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)$$

$$= \log(x^2+2x+5) + \tfrac{1}{2}\tan\frac{x+1}{2}.$$

The other general method employed for effecting integrations is what is known as "integration by parts," and is founded on the formula $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, whence

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx, \text{ or } \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Thus, to integrate $x\sin ax$, put $u = x$, $\frac{dv}{dx} = \sin ax$, and therefore $v = -\frac{1}{a}\cos ax$, in the formula, when

$$\int x\sin ax\,dx = -\frac{x}{a}\cos ax + \int \frac{\cos ax}{a}\,dx = -\frac{x}{a}\cos ax + \frac{\sin ax}{a^2};$$

similarly for $x^2\cos ax$, put $u = x^2$, $\frac{dv}{dx} = \cos ax$, and therefore $v = \frac{1}{a}\sin ax$, whence

$$\int x^2\cos ax\,dx = \frac{x^2}{a}\sin ax - \frac{2}{a}\int x\sin ax\,dx$$

$$= \frac{x^2}{a}\sin ax + \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax.$$

Again, to integrate $\left.\begin{matrix}\epsilon^{mx}\sin nx \\ \epsilon^{mx}\cos nx\end{matrix}\right\}$ and, put (1) $u=\epsilon^{mx}$, $\frac{dv}{dx}=\sin nx$, therefore $v=-\frac{1}{n}\cos nx$, or

$$\int \epsilon^{mx}\sin nx\,dx = -\epsilon^{mx}\frac{\cos nx}{n} + \frac{m}{n}\int \epsilon^{mx}\cos nx\,dx \ldots\ldots(A);$$

and (2) put $u=\sin nx$, $\frac{dv}{dx}=\epsilon^{mx}$, and therefore $v=\frac{1}{m}\epsilon^{mx}$, and we have

$$\int \epsilon^{mx}\sin nx\,dx = \epsilon^{mx}\frac{\sin nx}{m} - \frac{n}{m}\int \epsilon^{mx}\cos nx\,dx \ldots\ldots(B).$$

(A) and (B) together determine both integrals, subtract (B) from (A), and we get

$$\left(\frac{m}{n}+\frac{n}{m}\right)\int \epsilon^{mx}\cos nx\,dx = \epsilon^{mx}\left(\frac{\cos nx}{n}+\frac{\sin nx}{m}\right),$$

or
$$\int \epsilon^{mx}\cos nx\,dx = \epsilon^{mx}\frac{(m\cos nx + n\sin nx)}{m^2+n^2},$$

multiply (A) by $\frac{n}{m}$, (B) by $\frac{m}{n}$ and add, when we get

$$\left(\frac{m}{n}+\frac{n}{m}\right)\int \epsilon^{mx}\sin nx\,dx = \epsilon^{mx}\left(\frac{\sin nx}{n}-\frac{\cos nx}{m}\right);$$

$$\int \epsilon^{mx}\sin nx\,dx = \epsilon^{mx}\frac{(m\sin nx - n\cos nx)}{m^2+n^2}.$$

Another important integral $\int\sqrt{(a^2-x^2)}\,dx$ may be found by putting $u=\sqrt{(a^2-x^2)}$, $\frac{dv}{dx}=1$, and therefore $v=x$, then

$$\int\sqrt{(a^2-x^2)}\,dx = x\sqrt{(a^2-x^2)} - \int x.\frac{-x}{\sqrt{(a^2-x^2)}}\,dx$$

$$= x\sqrt{(a^2-x^2)} + \int\frac{x^2\,dx}{\sqrt{(a^2-x^2)}},$$

and

$$\int\frac{x^2\,dx}{\sqrt{(a^2-x^2)}} = \int\frac{a^2-(a^2-x^2)}{\sqrt{(a^2-x^2)}}\,dx = a^2\sin^{-1}\frac{x}{a} - \int\sqrt{(a^2-x^2)}\,dx;$$

therefore

$$\int \sqrt{(a^2 - x^2)}\, dx = x \sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\frac{x}{a} - \int \sqrt{(a^2 - x^2)}\, dx,$$

and therefore $\quad = \frac{1}{2}\left\{x \sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right\}.$

As this is one of the most frequently occurring integrals, it may be well to mention that the result may be obtained directly by considering the area of a circle, centre O, radius $= a$, P (fig. 5) any point of the circle, $OM = x$, $MP = y$, then $x^2 + y^2 = a^2$, or $MP = \sqrt{(a^2 - x^2)}$, and the area $AOMP = \int \sqrt{(a^2 - x^2)}\, dx$, corrected so as to vanish when x vanishes. But this area $=$ sector $AOP + \Delta POM$, $= \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} x \sqrt{(a^2 - x^2)}$, whence

$$\int \sqrt{(a^2 - x^2)}\, dx = \frac{1}{2}\left\{x \sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right\},$$

and no constant will be wanted if the integral vanish when x vanishes.

Another important integral $\int \sqrt{(2ax - x^2)}\, dx$ is at once reduced to this by writing it $\int \sqrt{\{a^2 - (x-a)^2\}}\, dx$, which is therefore $= \frac{1}{2}(x-a)\sqrt{\{a^2 - (x-a)^2\}} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right)$, but in this form the integral will not vanish when x vanishes, but when $x = a$; and as it is generally better to take the integral of such a form as to vanish with x, we must add such a constant to this as will make it do so. Now when $x = 0$, the above result becomes $-\frac{\pi a^2}{4}$, and therefore $\frac{\pi a^2}{4}$ should be added, and the result will be

$$\frac{x-a}{2} \sqrt{(2ax - x^2)} + \frac{a^2}{2}\left\{\frac{1}{2}\pi + \sin^{-1}\left(\frac{x-a}{a}\right)\right\},$$

or
$$\frac{x-a}{2} \sqrt{(2ax - x^2)} + \frac{a^2}{2} \operatorname{vers}^{-1}\left(\frac{x}{a}\right).$$

(For if $\frac{1}{2}\pi + \sin^{-1}\frac{x-a}{a} = \theta$, $\frac{x-a}{a} = \sin(\theta - \frac{1}{2}\pi) = -\cos\theta$, or $x = a(1 - \cos\theta) = a \operatorname{vers}\theta$).

This integral $\int\sqrt{(2ax-x^2)}\,dx$ may also be found directly from the area of the circle. C (fig. 6) the centre, radius $OC=a$, $OM=x$, $MP=y$, then

$$y^2+\overline{x-a}|^2=a^2, \text{ or } y=\sqrt{(2ax-x^2)};$$

therefore area $OMP=\int\sqrt{(2ax-x^2)}\,dx$, corrected so as to vanish when $x=0$, $=$ sector $OCP\pm\Delta CMP$; but whichever it be, the whole is $\frac{a^2}{2}\operatorname{vers}^{-1}\left(\frac{x}{a}\right)+\frac{x-a}{2}\sqrt{(2ax-x^2)}$, the second term being positive or negative correctly.

Integrals of the form

$$\int\frac{dx}{a+b\sin x},\ \int\frac{dx}{a+b\cos x},\ \int\frac{dx}{a+b\cos x+c\sin x}$$

can be reduced to known forms by the assumption $\tan\frac{1}{2}x=z$. Thus

$$\int\frac{dx}{\sin x}=\int\frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2}}=\int\frac{dz}{z}=\log z=\log\tan\tfrac{1}{2}x.$$

So $$\int\frac{dx}{\cos x}=\int\frac{2dz}{1+z^2}\div\left(\frac{1-z^2}{1+z^2}\right)$$

$$=\int\frac{2dz}{1-z^2}=\log\left(\frac{1+z}{1-z}\right)=\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x);$$

this may be found from the preceding by putting $\frac{1}{2}\pi+x$ instead of x, on both sides.

$$\int\frac{dx}{a+b\sin x}=\int\frac{2dz}{a(1+z^2)+2bz}$$

$$=2\int\frac{dz}{a\left(z+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a}}=\frac{2}{a}\int\frac{d\left(z+\frac{b}{a}\right)}{\left(z+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}}$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\frac{z+\frac{b}{a}}{\frac{\sqrt{(a^2-b^2)}}{a}}\right\}=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\frac{az+b}{\sqrt{(a^2-b^2)}}\right\},$$

$$\int \frac{dx}{a+b\cos x} = \int \frac{2dz}{a(1+z^2)+b(1-z^2)} = 2\int \frac{dz}{(a+b)+(a-b)z^2}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1}\left\{z\sqrt{\left(\frac{a-b}{a+b}\right)}\right\}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan \tfrac{1}{2}x\right\},$$

each of these will be of another form if $b > a$. The results given are, however, much the more frequently required, not that *these* should be remembered, but the *method;* viz. when $\sin x$ or $\cos x$ occur in the denominator in the first power, put $\tan \frac{1}{2}x = z$.

Should $b > a$ the results are

$$\int \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{(b^2-a^2)}} \log\left\{\frac{a\tan\frac{1}{2}x + b - \sqrt{(b^2-a^2)}}{a\tan\frac{1}{2}x + b + \sqrt{(b^2-a^2)}}\right\},$$

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{(b^2-a^2)}} \log\left\{\frac{\sqrt{(b+a)}+\sqrt{(b-a)}\tan\frac{1}{2}x}{\sqrt{(b+a)}-\sqrt{(b-a)}\tan\frac{1}{2}x}\right\}.$$

The remaining integral $\int \frac{dx}{a+b\sin x + c\cos x}$ should be changed by putting

$$b = m\cos\alpha, \quad c = m\sin\alpha,$$

$$\left\{\text{therefore } m = \sqrt{(b^2+c^2)}, \ \tan\alpha = \frac{c}{b}\right\},$$

and it becomes

$$\int \frac{dx}{a+m\sin(x+\alpha)} \quad \text{or} \quad \int \frac{d(x+\alpha)}{a+m\sin(x+\alpha)}.$$

The three integrals

$$\int \frac{dx}{a+b\sin^2 x}, \quad \int \frac{dx}{a+b\cos^2 x}, \quad \int \frac{dx}{a+b\cos^2 x + c\sin^2 x}$$

are essentially of the same form $\int \frac{dx}{m\cos^2 x + n\sin^2 x}$, and may be reduced by taking $\tan x = z$, so that the last integral becomes

$$\int \frac{dz}{m+nz^2} = \frac{1}{\sqrt{(mn)}} \tan^{-1}\left\{\sqrt{\left(\frac{n}{m}\right)} z\right\},$$

if m, n have the same sign, or

$$\frac{1}{2\sqrt{(-mn)}} \log \left\{\frac{\sqrt{(-n)}\,z + \sqrt{m}}{\sqrt{(-n)}\,z - \sqrt{m}}\right\},$$

if n be negative and m positive, or *vice versâ.*

The following integrals

$$\int \frac{x^m}{\sqrt{(a^2 - x^2)}}\, dx, \int \frac{x^m}{\sqrt{(2ax - x^2)}}\, dx, \int x^m \sqrt{(a^2 - x^2)}\, dx,$$

and $$\int x^m \sqrt{(2ax - x^2)}\, dx,$$

frequently occur with different integral values of m, and are to be obtained by what are called *formulæ of reduction;* *i.e.* in each case the integral is expressed in terms of another of the same form, but with a lower power of x, and this is carried on until the integral involved is known.

I. Take $P_m = \int \frac{x^m}{\sqrt{(a^2 - x^2)}}\, dx = -\int x^{m-1} \frac{d}{dx}\{\sqrt{(a^2 - x^2)}\}\, dx$

$$= -x^{m-1}\sqrt{(a^2 - x^2)} + (m-1)\int x^{m-2}\sqrt{(a^2 - x^2)}\, dx,$$

and $\int x^{m-2}\sqrt{(a^2 - x^2)}\, dx = \int x^{m-2} \frac{(a^2 - x^2)}{\sqrt{(a^2 - x^2)}}\, dx = a^2 P_{m-2} - P_m$,

therefore $\quad P_m = -x^{m-1}\sqrt{(a^2 - x^2)} + (m-1)(a^2 P_{m-2} - P_m)$,

or $\quad mP_m = -x^{m-1}\sqrt{(a^2 - x^2)} + (m-1)\, a^2 P_{m-2}$.

If the limits of the integral be 0 and a; or $-a$, a, the integrated part vanishes at both limits, and we have

$$\int_0^a \frac{x^m}{\sqrt{(a^2 + x^2)}}\, dx = \frac{m-1}{m}\, a^2 \int_0^a \frac{x^{m-2}\, dx}{\sqrt{(a^2 - x^2)}}.$$

II. Take $P_m = \frac{x^m}{\sqrt{(2ax - x^2)}}\, dx = \int x^{m-1} \frac{\{a - (a - x)\}}{\sqrt{(2ax - x^2)}}\, dx$

$$= aP_{m-1} - \int x^{m-1} \frac{d}{dx}\{\sqrt{(2ax - x^2)}\}\, dx,$$

and $$\int x^{m-1} \frac{d}{dx}\{\sqrt{(2ax - x^2)}\}\, dx$$

$$= x^{m-1}\sqrt{(2ax - x^2)} - (m-1)\int x^{m-2}\sqrt{(2ax - x^2)}\, dx$$

$$= x^{m-1}\sqrt{(2ax - x^2)} - (m-1)\int \frac{x^{m-2}(2ax - x^2)}{\sqrt{(2ax - x^2)}}\, dx$$

$$= x^{m-1}\sqrt{(2ax - x^2)} - (m-1)(2aP_{m-1} - P_m);$$

therefore

$$P_m = -x^{m-1}\sqrt{(2ax-x^2)} + aP_{m-1} + (m-1)(2aP_{m-1} - P_m),$$

or $$mP_m = -x^{m-1}\sqrt{(2ax-x^2)} + (2m-1)aP_{m-1};$$

therefore $$\int_0^{2a} \frac{x^m}{\sqrt{(2ax-x^2)}}\,dx = \frac{2m-1}{m}\,a\int_0^{2a} \frac{x^{m-1}dx}{\sqrt{(2ax-x^2)}}.$$

III. $P_m = \int x^m \sqrt{(a^2-x^2)}\,dx$

$$= -\tfrac{1}{3}\int x^{m-1}\frac{d}{dx}\{(a^2-x^2)^{\frac{3}{2}}\}\,dx$$

$$= -\tfrac{1}{3}x^{m-1}(a^2-x^2)^{\frac{3}{2}} + \frac{m-1}{3}\int x^{m-2}(a^2-x^2)\sqrt{(a^2-x^2)}\,dx$$

$$= -\tfrac{1}{3}x^{m-1}(a^2-x^2)^{\frac{3}{2}} + \frac{m-1}{3}(a^2P_{m-2} - P_m),$$

or $$(m+2)P_m = -x^{m-1}(a^2-x^2)^{\frac{3}{2}} + (m-1)a^2P_{m-2},$$

whence $$\int_0^a x^m\sqrt{(a^2-x^2)}\,dx = \frac{m-1}{m+2}\,a^2\int_0^a x^{m-2}\sqrt{(a^2-x^2)}\,dx.$$

IV. $P_m = \int_0 x^m\sqrt{(2ax-x^2)}dx = \int x^{m-1}\{a-(a-x)\}\sqrt{(2ax-x^2)}dx$

$$= aP_{m-1} - \tfrac{1}{3}\int x^{m-1}\frac{d}{dx}\{(2ax-x^2)^{\frac{3}{2}}\}\,dx$$

$$= aP_{m-1} - \tfrac{1}{3}x^{m-1}(2ax-x^2)^{\frac{3}{2}} + \frac{m-1}{3}\int x^{m-2}(2ax-x^2)\sqrt{(2ax-x^2)}dx$$

$$= -\tfrac{1}{3}x^{m-1}(2ax-x^2)^{\frac{3}{2}} + aP_{m-1} + \frac{m-1}{3}\{2aP_{m-1} - P_m\},$$

whence $(m+2)P_m = -x^{m-1}(2ax-x^2)^{\frac{5}{2}} + (2m+1)aP_{m-1}$,

and $$\int_0^{2a} x^m\sqrt{(2ax-x^2)}\,dx = \frac{2m+1}{m+2}\,a\int_0^{2a} x^{m-1}\sqrt{(2ax-x^2)}\,dx.$$

Another such integral may be given, which really includes these four; or, at least, from it they can all be easily deduced.

$$P_m = \int \sin^m x\,dx = -\int \sin^{m-1}x\,\frac{d}{dx}(\cos x)\,dx$$

$$= -\sin^{m-1}x\cos x + (m-1)\int \sin^{m-2}x\cos^2 x\,dx$$

$$= -\sin^{m-1}x\cos x + (m-1)\int \sin^{m-2}x(1-\sin^2 x)dx$$

$$= -\sin^{m-1}x\cos x + (m-1)(P_{m-2} - P_m);$$

whence $$mP_m = -\sin^{m-1}x\cos x + (m-1)P_{m-2},$$

so that $\qquad \displaystyle\int_0^{\frac{1}{2}\pi} \sin^m x dx = \frac{m-1}{m} \int_0^{\frac{1}{2}\pi} \sin^{m-2} x dx.$

Putting $\frac{1}{2}\pi - x$ for x in this result, we get

$$\int_0^{\frac{1}{2}\pi} \cos^m x dx = \frac{m-1}{m} \int_0^{\frac{1}{2}\pi} \cos^{m-2} x dx;$$

and we also see that $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^m x dx = \int_0^{\frac{1}{2}\pi} \cos^m x dx$, (as is obvious, since the elements of each are exactly the same, but in reverse order).

So also $\displaystyle\int_0^{\pi} \sin^n x dx = \int_0^{\frac{1}{2}\pi} \sin^n x dx + \int_{\frac{1}{2}\pi}^{\pi} \sin^n x dx$, and if in the latter term we put $\pi - x$ for x it becomes $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^n x dx$; whence $\displaystyle\int_0^{\pi} \sin^n x dx = 2\int_0^{\frac{1}{2}\pi} \sin^n x dx$; (as is obvious from first principles, since the elements of the integral from $\frac{1}{2}\pi$ to π, are the same as those from 0 to $\frac{1}{2}\pi$ in the reverse order). The formulæ of reduction I., II., III., IV. can be all deduced from the formula for $\int \sin^n x dx$ by putting $x = a \sin z$ in I. and III., and $x = 2a \sin^2 z$, in II. and IV.

A very useful result in finding the values of *definite* integrals is the following:

$\displaystyle\int_0^a \phi(x)\, dx = \int_0^a \phi(a-x)\, dx$, which is obtained at once by putting $a - x$ for x in the first member. It must be remembered that a *definite* integral is not a function of x at all, and that it is of absolutely no matter by what symbol we denote the variable in the subject operated upon. Thus we might get for the above

$$\int_0^a \phi(x)\, dx = \int_a^0 \phi(a-z)\,(-dz),$$

since when $x = a$, $a - x$ or $z = 0$, and when $x = 0$, $z = a$,

$$= \int_0^a \phi(a-z)\, dz,$$

but this is exactly the same thing as $\displaystyle\int_0^a \phi(a-x)\, dx$.

The result is obvious if we draw the two curves $y=\phi(x)$, $y=\phi(a-x)$, suppose $BPQDC$ (fig. 7), and $bpqDc$, then if $OM=x$, $MP=\phi(x)$, $Am=x$, therefore $Om=a-x$, then $mp=MP$, since $mp=\phi(a-Om)=\phi(x)=MP$. The area of the curve $BOAC=\int_0^a \phi(x)\,dx$, and that of $cOAb=\int_0^a \phi(a-x)\,dx$, and these areas are manifestly equal, the bounding curves being similar and equal.

The following is also sometimes useful:

$$\int_0^{2a} \phi(x)\,dx=\int_0^a \{\phi(x)+\phi(2a-x)\}\,dx;$$

for $\int_0^{2a} \phi(x)\,dx=\psi(2a)-\psi(0)=\psi(2a)-\psi(a)+\psi(a)-\psi(0)$

$$=\int_a^{2a} \phi(x)\,dx+\int_0^a \phi(x)\,dx,$$

and if in the first term we put $2a-z$ for x, it becomes $\int_a^0 \phi(2a-z)(-dz)$ or $\int_0^a \phi(2a-z)\,dz$, or $\int_0^a \phi(2a-x)\,dx$. This is also easily proved geometrically like the former.

As examples of the use to be made of such formula, take

$$\int_0^\pi xF(\sin x)\,dx=\int_0^\pi (\pi-x)\,F\{\sin(\pi-x)\}\,dx$$

$$=\pi\int_0^\pi F(\sin x)\,dx-\int_0^\pi F(\sin x)\,dx,$$

therefore $\int_0^\pi xF(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi F(\sin x)\,dx$.

Thus $$\int_0^\pi \frac{x\,dx}{1+\sin x}=\frac{\pi}{2}\int_0^\pi \frac{dx}{1+\sin x}=\pi,$$

and $$\int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x}=\frac{\pi}{2}\int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x}$$

$$=-\frac{\pi}{2}\int_0^\pi \frac{d(\cos x)}{1+\cos^2 x}=-\frac{\pi}{2}\tan^{-1}(\cos x)_0^\pi=\frac{\pi^2}{4}.$$

F

So also $\int_0^{\frac{1}{4}\pi} \log(1+\tan x)\,dx = \int_0^{\frac{1}{4}\pi} \log\left\{1+\tan\left(\frac{\pi}{4}-x\right)\right\} dx$

$$= \int_0^{\frac{1}{4}\pi} \log\left(1+\frac{1-\tan x}{1+\tan x}\right) dx = \int_0^{\frac{1}{4}\pi} \{\log 2 - \log(1+\tan x)\}\,dx$$

$$= \frac{\pi}{4}\log 2 - \int_0^{\frac{1}{4}\pi} \log(1+\tan x)\,dx,$$

or $$\int_0^{\frac{1}{4}\pi} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2.$$

The following makes use of both formulæ,

let $$u = \int_0^{\frac{1}{2}\pi} \log(\sin x)\,dx,$$

then $$u = \int_0^{\frac{1}{2}\pi} \log(\cos x)\,dx \text{ by } (1);$$

therefore $$2u = \int_0^{\frac{1}{2}\pi} (\log\sin x + \log\cos x)\,dx,$$

or $$2u = \int_0^{\frac{1}{2}\pi} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\frac{1}{2}\pi} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2.$$

But $$\int_0^{\frac{1}{2}\pi} \log\sin 2x\,dx = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(\sin 2x)\,d(2x)$$

$$= \tfrac{1}{2}\int_0^{\pi} \log(\sin x)\,dx,$$

and therefore $$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \{\log\sin x + \log\sin(\pi - x)\}\,dx \text{ by } (2)$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (2\log\sin x)\,dx = u;$$

therefore $$2u = u - \frac{\pi}{2}\log 2, \text{ or } u = \frac{\pi}{2}\log(\tfrac{1}{2});$$

and we see by the proof that also

$$\int_0^{\frac{1}{2}\pi} \log(\cos x)\,dx = \frac{\pi}{2}\log(\tfrac{1}{2}); \text{ and } \int_0^{\pi} \log(\sin x)\,dx = \pi\log(\tfrac{1}{2}).$$

The most useful application of (2) is to such as the following:

$$\int_0^{\pi} F(\sin x)\,dx = 2\int_0^{\frac{1}{2}\pi} F(\sin x)\,dx;$$

$$\int_0^{\pi} F(\cos x)\,dx = \int_0^{\frac{1}{2}\pi} \{F(\cos x) + F(-\cos x)\}\,dx,$$

and of (1) $\displaystyle\int_0^{\frac{1}{2}\pi} F(\sin x)\,dx = \int_0^{\frac{1}{2}\pi} F(\cos x)\,dx.$

Thus

$$\int_0^{\frac{1}{2}\pi} \sin^2 x dx = \int_0^{\frac{1}{2}\pi} \cos^2 x dx = \int_0^{\frac{1}{2}\pi} (1-\sin^2 x) dx = \tfrac{1}{2}\pi - \int_0^{\frac{1}{2}\pi} \sin^2 x dx,$$

therefore $$\int_0^{\frac{1}{2}\pi} \sin^2 x dx = \tfrac{1}{4}\pi.$$

Integrating by parts

$$\int_0^{\frac{1}{2}\pi} (\log \sin x)\,dx$$

$$= (x \log \sin x)_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} x \cot x dx = -\int_0^{\frac{1}{2}\pi} \frac{x}{\tan x}\,dx,$$

the former vanishing at both limits; therefore

$$\int_0^{\frac{1}{2}\pi} \frac{x}{\tan x}\,dx = \tfrac{1}{2}\pi \log 2$$

$$= \left(\frac{x^2}{2 \tan x}\right)_0^{\frac{1}{2}\pi} + \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{x^2}{\sin^2 x}\,dx = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{x^2}{\sin^2 x}\,dx;$$

therefore $$\int_0^{\frac{1}{2}\pi} \frac{x^2}{\sin^2 x}\,dx = \pi \log 2.$$

The integral $\displaystyle\int_0^{\frac{1}{2}\pi} \log(\sin x)\,dx$ may be evaluated thus:

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(\sin^2 x)\,dx = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(\cos^2 x)\,dx$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(1 - \sin^2 x)\,dx$$

$$= -\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \{\sin^2 x + \tfrac{1}{2}\sin^4 x + \tfrac{1}{3}\sin^6 x + \dots\}$$

$$= -\tfrac{1}{2}\cdot\tfrac{1}{2}\pi\left\{\frac{1}{2} + \frac{1}{2}\cdot\frac{1.3}{2.4} + \frac{1}{3}\cdot\frac{1.3.5}{2.4.6} + \dots\right\}.$$

But $$\frac{1}{\sqrt{(1-x^2)}} = 1 + \tfrac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \dots;$$

therefore $\dfrac{1}{x\sqrt{(1-x^2)}} - \dfrac{1}{x} = \dfrac{x}{2} + \dfrac{1.3}{2.4}x^3 + \dfrac{1.3.5}{2.4.6}x^5 + \ldots;$

therefore

$$\int_0^1 \left\{\frac{dx}{x\sqrt{(1-x^2)}} - \frac{dx}{x}\right\} = \frac{1}{2}\left\{\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{3} + \ldots\right\}$$

$$= \int_0^{\frac{1}{2}\pi}\left(\frac{d\theta}{\sin\theta} - \frac{\cos\theta d\theta}{\sin\theta}\right) = \int_0^{\frac{1}{2}\pi} \tan\frac{\theta}{2}\,d\theta$$

$$= 2\log\left(\sec\frac{\theta}{2}\right)_0^{\frac{1}{2}\pi} = 2\log\sqrt{(2)} = \log 2.$$

Questions on Integral Calculus. I.

1. If $\psi'(x) = \phi(x)$, and $\dfrac{dU}{dx} = \phi(x)$, prove that $U = \psi(x) + C$ where C may have any value independent of x. Explain what is meant by an *indefinite*, a *corrected*, and a *definite* integral.

2. If $\psi'(x) = \phi(x)$, and $h = \dfrac{1}{n}(b-a)$, prove that the *limit* of the sum of the n elements

$$h[\phi(a) + \phi(a+h) + \phi(a+2h) + \ldots + \phi\{a+(n-1)h\}],$$

when $h=0$, is $\psi(b) - \psi(a)$. Hence prove that the sum

$$\frac{1}{n} + \frac{1}{\sqrt{(n^2-1^2)}} + \frac{1}{\sqrt{(n^2-2^2)}} + \ldots + \frac{1}{\sqrt{\{n^2-(n-1^2)\}}}$$

tends to the limit $\frac{1}{2}\pi$ when $n = \infty$. Find the area of the curve $y^2 = 4ax$ included between the ordinates $x = 2a$, $x = 8a$, and the curve.

3. Write down the integrals of x^m, $\dfrac{1}{x}$, a^x, ε^{mx}, $\sin x$, $\cos x$, $\sec^2 x$, $\dfrac{1}{\sqrt{(a^2-x^2)}}$, $\dfrac{1}{a^2+x^2}$, $\dfrac{1}{\sqrt{(x^2 \pm a^2)}}$; also deduce the value of $\displaystyle\int\frac{dx}{x}$ from the equation $\displaystyle\int x^m\,dx = \frac{x^{m+1}}{m+1} + C.$

4. If $x=f(z)$, prove that $\int\phi(x)\,dx=\int\phi\{f(z)\}.f'(z)\,dz$; and obtain the integrals $\int\frac{dx}{\sqrt{(2ax-x^2)}}$, and $\int\frac{dx}{x\sqrt{(x^2-a^2)}}$, by putting (1) $x=a(1-\cos z)$, (2) $x=a\sec z$.

5. Prove the formula for integration by parts,

$$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx.$$

Apply it to find the integrals of $x\cos ax$, $x^2\sin ax$, $x^n\log x$, $\varepsilon^{mx}\cos nx$, and $\cos ax\cos bx$.

6. *Shew how* to integrate

$$\frac{1}{\sqrt{(a+bx\pm cx^2)}},\quad \frac{1}{a+bx+cx^2},\ \text{and}\ \frac{1}{(x+a)\sqrt{(b+cx+ex^2)}}.$$

7. Integrate

$$\frac{1}{a+b\cos x},\quad \frac{1}{a+b\sin x},\quad \frac{1}{a+b\cos x+c\sin x},$$

(when $a>b$ in the first two and $a^2>b^2+c^2$ in the third), also integrate $\frac{1}{\cos x}$, $\frac{1}{\sin x}$, $\frac{1}{\cos x+\sin x}$; *or* deduce them from the former.

8. Prove that $\int_0^a\sqrt{(a^2-x^2)}\,dx=\frac{1}{4}\pi a^2$,

$$\int_0^{2a}\sqrt{(2ax-x^2)}\,dx=\tfrac{1}{2}\pi a^2=2\int_0^a\sqrt{(2ax-x^2)}\,dx,$$

and

$$\int_0^{2a}x\sqrt{(2ax-x^2)}\,dx=\tfrac{1}{2}\pi a^3.$$

9. Prove that $\int_0^1 x^r\log x\,dx=\frac{-1}{(r+1)^2}$,

$$\int_0^1 x^{r-1}(\log x)^2\,dx=\frac{2}{r^3};\quad \int_0^1 x^{r-1}(\log x)^n\,dx=\frac{(-1)^n\lfloor n}{r^{n+1}};$$

and deduce

$$\int_0^1\frac{\log x}{x-1}\,dx=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\text{ to }\infty\ \left(\text{or }\frac{\pi^2}{6}\right);$$

$$\int_0^1\left(\frac{\log x}{1-x}\right)^2dx=\tfrac{1}{2}\int_0^\infty\left(\frac{\log x}{1-x}\right)^2dx=\tfrac{1}{3}\pi^2.$$

10. Prove that $\int_0^{\frac{1}{2}\pi} \frac{dx}{1+e\cos x} = \frac{\cos^{-1} e}{\sqrt{(1-e^2)}}$;
and

$$\int_0^{\frac{1}{2}\pi} \frac{dx}{(1+e\cos x)^2} = \frac{-e}{1-e^2} + \frac{1}{(1-e^2)^{\frac{3}{2}}}\cos^{-1}(e),\ (e<1).$$

$\left(\text{To integrate } \frac{1}{(1+e\cos x)^2} \text{ first differentiate } \frac{\sin x}{1+e\cos x}\right)$.

Questions on Integral Calculus. II.

1. Find the value of

$$\int_1^2 \sqrt{(-2+3x-x^2)}\,dx,$$

generally of $\int_a^b \sqrt{\{-ab+(a+b)x-x^2\}}\,dx$,

and of $\int_a^b x\sqrt{\{-ab+(a+b)x-x^2\}}\,dx$.

$\Big\{$Put $x = \frac{a+b}{2} + z$,

the answers are $\frac{1}{8}\pi$, $\frac{1}{8}\pi(b-a)^2$, $\frac{\pi}{16}(b+a)(b-a)^2\Big\}$.

Prove also that

$$\int_0^{2n} x(x-1)(x-2)(x-3)\dots(x-2n)\,dx = 0,\ (\text{put } x = n+z).$$

2. Prove the following formulæ of reduction:

(1) $\int_a^a x^n \sqrt{(a^2-x^2)}\,dx = \frac{n-1}{n+2}a^2 \int_0^a x^{n-2}\sqrt{(a^2-x^2)}\,dx$;

(2) $\int_0^{2a} x^n \sqrt{(2ax-x^2)}\,dx = \frac{2n+1}{n+2}a\int_0^{2a} x^{n-1}\sqrt{(2ax-x^2)}dx$;

(3) $\int_0^{\frac{1}{2}\pi} \sin^n x dx = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi} \sin^{n-2} x dx$

$$= \tfrac{1}{2}\int_0^{\pi} \sin^n x dx = \int_0^{\frac{1}{2}\pi} \cos^n x dx.$$

3. Assuming that the limiting value of the ratio

$$\int_0^{\frac{1}{2}\pi} \sin^{2n} x dx : \int_0^{\frac{1}{2}\pi} \sin^{2n+1} x dx,$$

when n is indefinitely increased, is one of equality, prove Wallis' Formula

$$\frac{\pi}{2} = \frac{2^2}{2^2-1} \frac{4^2}{4^2-1} \frac{6^2}{6^2-1} \cdots \frac{(2n)^2}{(2n)^2-1}, \ n = \infty.$$

4. Prove that $\displaystyle\int_0^\pi \frac{dx}{1+e\cos x} = \frac{\pi}{\sqrt{(1-e^2)}}$,

$$\int_0^\pi \frac{dx}{1+e\cos x)^2} = \frac{\pi}{(1-e^2)^{\frac{3}{2}}}, \int_0^\pi \frac{dx}{(1+e\cos x)^3} = \frac{\pi\left(1+\frac{e^2}{2}\right)}{(1-e^2)^{\frac{5}{2}}};$$

and generally that

$$\int_0^\pi \frac{dx}{(1+e\cos x)^{n+1}} = \frac{\pi}{(1-e^2)^{n+\frac{1}{2}}} \int_0^\pi (1+e\cos x)^n dx$$

$$= \frac{\pi}{(1-e^2)^{n+\frac{1}{2}}} \left\{1 + \frac{n(n-1)}{2^2} e^2 + \frac{n(n-1)(n-2)(n-3)}{2^2.4^2} e^4 + \ldots\right\},$$

$\left\{\text{transformed by putting } \tan\frac{1}{2}x \tan\frac{1}{2}z = \sqrt{\left(\frac{1+e}{1-e}\right)}\right\}$.

5. Prove that $\displaystyle\int_0^a \phi(x)\, dx = \int_0^a \phi(a-x)\, dx$,

and thence that

$$\int_0^\pi x \sin^n x dx = \tfrac{1}{2}\pi \int_0^\pi \sin^n x dx = \pi \int_0^{\frac{1}{2}\pi} \sin^n x dx,$$

and that
$$\int_0^\pi \frac{x dx}{1+\sin x} = \pi.$$

6. Prove that $\displaystyle\int_0^\pi \frac{dx}{1+e\sin x} = \frac{1}{\sqrt{(1-e^2)}} \cos^{-1}(e)$,

and deduce

$$\int_0^\pi \frac{\log(1+e\sin x)}{\sin x} dx = \tfrac{1}{2}\alpha(\pi-\alpha), \text{ where } \sin\alpha = e.$$

7. Obtain formulæ of reduction for

$$\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}} dx, \int_0^{2a} \frac{x^n}{\sqrt{(2ax-x^2)}} dx.$$

If $$P_n = \int_1^2 x^n \sqrt{(-2+3x-x^2)}\, dx,$$

prove that $2(n+2)P_n - 3(2n-1)P_{n-1} + 4(n-1)P_{n-2} = 0$.

8. Obtain the complete areas of the ellipses,

$$(1)\quad ax^2 + 2hxy + by^2 = 1,$$

$$(2)\quad ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0.$$

For any value of x, we have, in (1),

$$by = -hx \pm \sqrt{\{b - (ab - h^2)x^2\}},$$

whence, if y_1, y_2 be these two values of y, the element of the integral is $(y_2 - y_1)\,dx$, and this must extend over the values of x for which y_1 and y_2 continue possible; hence, if we put $b = m^2(ab - h^2)$, the limits of x are $-m$, m, and the

$$\text{area} = \frac{2}{b}\int_{-m}^{m} \sqrt{\{b - (ab-h^2)x^2\}}\, dx = \frac{2}{b}\sqrt{(ab-h^2)}\int_{-m}^{m}\sqrt{(m^2-x^2)}\, dx$$

$$= \frac{2}{b}\sqrt{(ab-h^2)}\,\frac{\pi}{2}\,m^2, \text{ or } \frac{\pi}{\sqrt{(ab-h^2)}};$$

(2) treated in exactly the same way, gives the area

$$\pi(af^2 + bg^2 + ch^2 - abc - 2fgh) \div (ab - h^2)^{\frac{3}{2}}.$$

9. Obtain the whole area of the loop of the curve $y^2 = x^2\,\dfrac{a-x}{a+x}$; also of the area between the curve and the asymptote. (1) $\pi a^2\left(2 - \dfrac{\pi}{2}\right)$, (2) $\pi a^2\left(2 + \dfrac{\pi}{2}\right)$.

Differentiation of a Definite Integral with respect to some Symbol involved.

Suppose $u = \int_a^b \phi(x, z)\, dx$, where z is a quantity independent of x, and also of a and b, then if

$$\frac{d}{dx}\{\psi(x, z)\} = \phi(x, z), \quad u = \psi(b, z) - \psi(a, z).$$

In the case where a, b are independent of z, to find $\frac{du}{dz}$, we shall have

$$u + \Delta u = \int_a^b \phi(x, z + \Delta z) dx;$$

therefore $$\Delta u = \int_a^b \{\phi(x, z + \Delta z) - \phi(x, z)\} dx,$$

$$\frac{\Delta u}{\Delta z} = \int_a^b \frac{\phi(x, z + \Delta z) - \phi(x, z)}{\Delta z} dx;$$

hence, taking the limit, we shall have

$$\frac{du}{dz} = \int_a^b \frac{d}{dz}\{\phi(x, z)\} dx;$$

or we differentiate the quantity under the integral sign with respect to z, just as if no integral sign existed. Of course in exactly the same way we may integrate both sides with respect to z, when a, b are independent of z. If, however, a, b are related to z in any way, the complete differential coefficient with respect to z will consist of three parts, (1) that which we have already formed, (2) that arising from the variation of (b), (3) that arising from the variation of a. The first we already have, the second

$$= \frac{du}{db}\frac{db}{dz} = \frac{d}{db}\{\psi(b, z)\}\frac{db}{dz} = \phi(b, z)\frac{db}{dz};$$

since $$\frac{d}{dx}\{\psi(x, z)\} \equiv \phi(x, z);$$

similarly $$\frac{du}{da}\frac{da}{dz} = -\phi(a, z)\frac{da}{dz};$$

or the complete differential coefficient of u with respect to z is

$$\int_a^b \frac{d}{dz}\{\phi(x, z)\} dx + \phi(b, z)\frac{db}{dz} - \phi(a, z)\frac{da}{dz}.$$

Thus $$\int_a^{4a} \sqrt{(x)}\, dx = \tfrac{2}{3}a^{\frac{3}{2}}(8 - 1) = \tfrac{14}{3}a^{\frac{3}{2}};$$

and the differential coefficient of this with respect to a will be (1) $0+(2)$, $\sqrt{(4a)}\,4-(3)$, $\sqrt{(a)}$ or $7\sqrt{(a)}$ which is obviously true.

If, however, we had taken $\int_a^{4a} \sqrt{(ax)}\, dx = \frac{14}{3} a^2$, the complete differential coefficient with respect to a is

$$\int_a^{4a} \frac{\sqrt{(x)}}{2\sqrt{(a)}}\, dx + \sqrt{(4a^2)}\, 4 - \sqrt{(a^2)},$$

or $$\tfrac{1}{3}\left\{\frac{(4a)^{\frac{3}{2}} - a^{\frac{3}{2}}}{\sqrt{(a)}}\right\} + 8a - a, \text{ or } a\left(\tfrac{7}{3}+7\right), \text{ or } \frac{28a}{3},$$

which is obviously correct.

The second differential coefficient is often calculated, but it is better to use the above again when necessary. A good example of such *integration* is the following: we have

$$\int_0^{\pi} \frac{dx}{1+e\sin x} = 2\int_0^{\infty} \frac{dz}{1+z^2+2ez}$$

$$= 2\int_0^{\infty} \frac{dz}{1-e^2+(z+e^2)} = \frac{2}{\sqrt{(1-e^2)}} \tan^{-1}\left\{\frac{z+e}{\sqrt{(1-e^2)}}\right\}_0^{\frac{1}{2}\pi}$$

$$= \frac{2}{\sqrt{(1-e^2)}}\left[\tfrac{1}{2}\pi - \tan^{-1}\left\{\frac{e}{\sqrt{(1-e^2)}}\right\}\right]$$

$$= \frac{2}{\sqrt{(1-e^2)}}\left(\tfrac{1}{2}\pi - \sin^{-1} e\right) = \frac{2}{\sqrt{(1-e^2)}} \cos^{-1}(e);$$

whence integrating both sides with respect to e, which has no concern with the limits,

$$\int_0^{\pi} \frac{\log(1+e\sin x)}{\sin x}\, dx = -\{\cos^{-1}(e)\}^2 + K,$$

K being independent of e, and when $e=0$, the left hand vanishes, therefore $K = (\frac{1}{2}\pi)^2$, whence if we put $e = \sin\theta$, we have

$$\int_0^{\pi} \frac{\log(1+\sin\theta\sin x)}{\sin x}\, dx = (\tfrac{1}{2}\pi)^2 - (\tfrac{1}{2}\pi-\theta)^2 = \pi\theta - \theta^2;$$

θ being an angle less than $\frac{1}{2}\pi$, since we take $\theta=0$, when $e=0$. Similarly

$$\int_0^\pi \frac{\log(1-\sin\theta\sin x)}{\sin x}\,dx = -\pi\theta - \theta^2,$$

whence $$\int_0^\pi \frac{1}{\sin x}\log\left(\frac{1+\sin\theta\sin x}{1-\sin\theta\sin x}\right)dx = 2\pi\theta;$$

and also $$\int_0^{\frac{1}{2}\pi} \frac{1}{\sin x}\log\left(\frac{1+\sin\theta\sin x}{1-\sin\theta\sin x}\right)dx = \pi\theta,$$

and $$\int_0^\pi \frac{x}{\sin x}\log\left(\frac{1+\sin\theta\sin x}{1-\sin\theta\sin x}\right)dx = \pi^2\theta.$$

Also $$\int_0^\pi \frac{1}{\sin x}\log(1-\sin^2\theta\sin^2 x)\,dx = -2\theta^2$$

$$= 2\int_0^{\frac{1}{2}\pi} \frac{dx}{\sin x}\log(1-\sin^2\theta\sin^2 x)\,dx,$$

and $$\int_0^\pi \frac{x}{\sin x}\log(1-\sin^2\theta\sin^2 x)\,dx$$

$$= \tfrac{1}{2}\pi\int_0^\pi \frac{dx}{\sin x}\log(\ldots) = -\pi\theta^2.$$

Examples of the Use of this Method.

1. Determine the value of $\int_0^{\frac{1}{4}\pi} \tan^{-1}\{m\sqrt{(1-\tan^2 x)}\}\,dx$.

2. If a string just surround a closed oval curve, and another curve be formed by unwinding this string, beginning at a point P, and unwinding the whole, the whole arc of this *involute* will be a maximum or minimum when P is such a point that the perimeter of the circle of curvature there is equal to the perimeter of the oval, a maximum if unwound in the direction in which the curvature decreases from P, and a minimum for the opposite direction.

3. A string of given length is fixed to a curve at O, and laid along the arc OP of the curve, it is then unwound and bound on to the curve on the other side of O along the arc OP'. Prove that the arc traced out by the end

of the string will be a maximum or minimum when the tangents at P, P' are equally inclined to that at O; and that it will be a maximum or minimum according as the curvature at O is greater or less than the arithmetic mean between the curvatures at P, P'.

4. A family of curves $r = cf(\theta)$ is described, the pole being a point O, and any one of these curves meets the prime radius in A, and a fixed circle with its centre at O and radius a in P, prove that when the area AOP is a maximum or minimum, it is equal to half the triangle POT.

I give solutions of (2), (3), (4), which I think may be instructive to a student.

(2) O any *fixed* point on the curve, $OP = c$, $OQ = s$, α; θ the angles which the tangents at P, Q make with that at O (fig. 8), then if S be the whole arc of the involute

$$S = \int_{\alpha}^{2\pi+\alpha} (s - c)\, d\theta;$$

therefore

$$\frac{dS}{d\alpha} = -\frac{dc}{d\alpha}(2\pi) + (PQ)_{\alpha}^{2\pi+\alpha} = \text{perimeter of oval} - 2\pi\frac{dc}{d\alpha},$$

or S is a maximum or minimum if $2\pi \times \dfrac{dc}{d\alpha}$ = perimeter of oval $\left(\text{and } \dfrac{dc}{d\alpha} = \text{radius of curvature at } P\right)$, a maximum if $\dfrac{dc}{d\alpha}$ increases from P towards Q, *i.e.* if the curvature decreases. If unwound in the opposite direction from P, the arc of the involute $= \displaystyle\int_{\alpha}^{2\pi+\alpha} (P - s + c)\, d\theta$, where P is the perimeter, and the sum of the two $= 2\pi P$, or is the same for every point, so that if one be a maximum the other must be a minimum.

(3) The same notation, arc $QP = \text{arc} PQ' = a$, β, β' the angles between the tangents at Q, Q' (fig. 9), and that at P, then the whole arc generated

$$= \pi a + \int_{\alpha-\beta}^{\alpha} (s + a - c)\, d\theta + \int_{\alpha}^{\alpha+\beta'} (c + a - s)\, d\theta = u,$$

therefore $$\frac{du}{d\alpha} = -\beta \frac{dc}{d\alpha} + \beta' \frac{dc}{d\alpha} + a - a,$$

$$= (\beta' - \beta) \frac{dc}{d\alpha} = (\beta' - \beta) \times \text{radius of curvature at } P,$$

$$\frac{d^2u}{d\alpha^2} = \left(\frac{d\beta'}{d\alpha} - \frac{d\beta}{d\alpha}\right) \rho, \text{ when } \beta' = \beta,$$

and $$\frac{d(\alpha - \beta)}{d(c - a)} = \text{curvature at } Q = \frac{1}{r_1} = \frac{d\alpha}{dc}\left(1 - \frac{d\beta}{d\alpha}\right);$$

therefore $$1 - \frac{d\beta}{d\alpha} = \frac{\rho}{r_1}, \quad 1 + \frac{d\beta'}{d\alpha} = \frac{\rho}{r_2},$$

or $$\frac{d^2u}{d\alpha^2} = \rho^2\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{\rho}\right);$$

therefore for a maximum

$$\beta' = \beta, \text{ and } \frac{1}{r_1} + \frac{1}{r_2} < \frac{2}{\rho}.$$

For a minimum $\beta' = \beta$, and $\dfrac{1}{r_1} + \dfrac{1}{r_2} > \dfrac{2}{\rho}$.

(4) $$u = \tfrac{1}{2}\int_0^\alpha c^2 \overline{f\theta}|^2 \, d\theta, \text{ where } cf(\alpha) = a,$$

therefore $$f'(\alpha) = -\frac{a}{c^2}\frac{dc}{d\alpha};$$

therefore $$\frac{du}{d\alpha} = c\frac{dc}{d\alpha}\int_0^\alpha \overline{f\theta}|^2 \, d\theta + \tfrac{1}{2}c^2\overline{f(\alpha)}|^2$$

$$= \tfrac{1}{2}c^2\overline{f\alpha}|^2 + 2\frac{u}{c}\frac{dc}{d\alpha} = \tfrac{1}{2}\overline{cf\alpha}|^2 - 2u\frac{c}{a}f'(\alpha).$$

Now $cf(\alpha) = a$, and $\dfrac{f'(\alpha)}{f(\alpha)} = \cot OPT$ (fig. 10); therefore for a maximum or minimum

$$a = 4\frac{u}{a}\cot OPT, \text{ or } u = \tfrac{1}{4}a^2 \tan OPT,$$
$$= \tfrac{1}{2}\Delta POT.$$

Differential Calculus.

Successive Differentiation.

Having investigated methods of finding the differential coefficients of any functions of x, we next consider rules for the second, third, ... differential coefficients. We have denoted the differential coefficient of y by $\frac{d}{dx}y$, *i.e.* we may say that we denote the operation of differentiating with respect to x by the symbol $\frac{d}{dx}$, and a natural extension of the same notation will lead us to denote the operation of differentiating 2, 3, ... n times by the symbols $\left(\frac{d}{dx}\right)^2$, $\left(\frac{d}{dx}\right)^3$, ... $\left(\frac{d}{dx}\right)^n$, or the second, third, ... n^{th} differential coefficients of y by $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... $\frac{d^ny}{dx^n}$. This notation has the great advantage that the indices denoting the number of operations follow the laws of indices in ordinary algebra; *i.e.* $\left(\frac{d}{dx}\right)^p \left(\frac{d}{dx}\right)^q y = \left(\frac{d}{dx}\right)^{p+q} y$, and any equation which holds in ordinary algebra between symbols of quantities will hold also when members of the equation are these symbols of operation. The most important formula of successive differentiation is at once found in this manner; we know that if u, v be two different functions of x,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

or, we may write it, the operation $\frac{d}{dx}$ applied to the product uv is equivalent to the sum of the operations D_1 and D_2, where D_1 represents the operation of differentiating u only, and D_2 that of differentiating v only. Hence the operation

$$\left(\frac{d}{dx}\right)^n = (D_1 + D_2)^n,$$

or $\frac{d^n}{dx^n}(uv)=\left\{D_1^{\,n}+nD_1^{\,n-1}D_2+\frac{n(n-1)}{\lfloor 2}D_1^{\,n-2}D_2^{\,2}+\ldots+D_2^{\,n}\right\}uv,$

but $$D_1^{\,n}(uv)=v\frac{d^nu}{dx^n},$$

since D_1 operates on u only,

$$D_1^{\,n-1}D_2(uv)=\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}}, \text{ and so on,}$$

or $$\frac{d^n(uv)}{dx^n}=v\frac{d^nu}{dx^n}+n\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}}$$
$$+\frac{n(n-1)}{\lfloor 2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}}+\ldots+u\frac{d^nv}{dx^n}.$$

Of course the same holds for the product of three or more factors, the coefficients being those of the multinomial theorem.

Thus $$\frac{d^3(uvw)}{dx^3}=vw\frac{d^3u}{dx^3}+wu\frac{d^3v}{dx^3}+uv\frac{d^3w}{dx^3}$$
$$+3w\frac{d^2u}{dx^2}\frac{dv}{dx}+5\text{ terms of the same form }+6\frac{du}{dx}\frac{dv}{dx}\frac{dw}{dx},$$

and in general

$$\frac{d^n(uvw)}{dx^n}=(D_1+D_2+D_3)^n\,uvw,$$

D_1 operating on u only, D_2 on v, D_3 on w, but each meaning $\frac{d}{dx}$.

The n^{th} differential coefficients of the functions $\sin x$, $\cos x$, $\log x$ are readily found.

Thus $$\frac{d}{dx}(\sin x)=\cos x=\sin(x+\tfrac{1}{2}\pi);$$

therefore

$$\frac{d^2(\sin x)}{dx^2}=\frac{d}{dx}\{\sin(x+\tfrac{1}{2}\pi)\}=\cos(x+\tfrac{1}{2}\pi)=\sin\left(x+\frac{2\pi}{2}\right),$$

and so on, each differentiation adding $\frac{1}{2}\pi$ to the angle, or

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right);$$

$\frac{d^n}{dx^n}(\cos x)$ in exactly the same way $= \cos\left(x + \frac{n\pi}{2}\right)$, or may be deduced from the last, for since

$$\frac{d}{dx}(\sin x) = \cos x,$$

therefore

$$\frac{d^n}{dx^n}(\cos x) = \frac{d^{n+1}(\sin x)}{dx^{n+1}} = \sin\{x + (n+1)\tfrac{1}{2}\pi\} = \cos\left(x + \frac{n\pi}{2}\right);$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}, \quad \frac{d^2}{dx^2}(\log x) = -\frac{1}{x^2}, \quad \frac{d^3}{dx^3}(\log x) = \frac{1.2}{x^3},$$

and so on; therefore

$$\frac{d^n}{dx^n}(\log x) = (-1)^{n-1}\frac{\lfloor n-1}{x^n}.$$

The n^{th} differential coefficients of $\sin(x+\alpha)$, $\cos(x+\alpha)$, $\log(x+c)$ may be at once written down from the above, viz.

$$\sin\left(x + \alpha + \frac{n\pi}{2}\right), \quad \cos\left(x + \alpha + \frac{n\pi}{2}\right), \quad \frac{(-1)^{n-1}\lfloor n-1}{(x+c)^n},$$

and
$$\frac{d^n}{dx^n}\sin(ax) = a^n \sin\left(ax + \frac{n\pi}{2}\right),$$

$$\frac{d^n}{dx^n}\cos(ax) = a^n \cos\left(ax + \frac{n\pi}{2}\right).$$

Since
$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}, \quad \cos^3 x = \frac{\cos 3x + 3\cos x}{4}, \text{ \&c.},$$

the n^{th} differential coefficients of these and all like functions can be at once obtained.

The n^{th} differential coefficient of (xu) is

$$x\frac{d^n u}{dx^n} + n\frac{d^{n-1}u}{dx^{n-1}}, \text{ since } \frac{d^2}{dx^2}(x) = 0, \text{ \&c.},$$

the n^{th} differential coefficient of (x^2u) is

$$x^2\frac{d^n u}{dx^n}+2nx\frac{d^{n-1}u}{dx^{n-1}}+n(n-1)\frac{d^{n-2}u}{dx^{n-2}},$$

for the like reason. In general, if one of the functions be a rational algebraical function of r dimensions, the n^{th} differential coefficient $(n>r)$ will only contain $(r+1)$ terms, since the $(r+1)^{\text{th}}$ and all subsequent differential coefficients of this function will vanish.

The n^{th} differential coefficient of

$$\frac{1}{a-x} \text{ is } \frac{\lfloor n}{(a-x)^{n+1}}, \text{ and of } \frac{1}{a+bx} \text{ is } \frac{(-1)^n b^n \lfloor n}{(a+bx)^{n+1}}.$$

We can find the n^{th} differential coefficient of $\dfrac{ax+b}{x^2-c^2}$ by separating it into the sum of two fractions, with denominators $x-c$, $x+c$, and so with any algebraical fraction whose denominator is the product of simple factors; of course all are so, but if the factors be unreal, we shall have to use De Moivre's Theorem to get the result in a real form. Thus, if $y=\dfrac{1}{x^2-x+1}$, the factors of x^2-x+1 are $x-\alpha$, $x-\beta$, where

$$\alpha=\cos\tfrac{1}{3}\pi+\sqrt{(-1)}\sin\tfrac{1}{3}\pi,\quad \beta=\cos\tfrac{1}{3}\pi-\sqrt{(-1)}\sin\tfrac{1}{3}\pi;$$

and if we assume $y=\dfrac{A}{x-\alpha}+\dfrac{B}{x-\beta}$ so that

$$1\equiv A(x-\beta)+B(x-\alpha),$$

we have
$$A=\frac{1}{\alpha-\beta}=-B,$$

or $$y=\frac{1}{\alpha-\beta}\left\{\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right\}=\frac{1}{\beta-\alpha}\left\{\frac{1}{\alpha-x}-\frac{1}{\beta-x}\right\};$$

therefore $$\frac{d^n y}{dx^n}=\frac{\lfloor n}{\beta-\alpha}\left\{\frac{1}{(\alpha-x)^{n+1}}-\frac{1}{(\beta-x)^{n+1}}\right\}$$

$$=\frac{\lfloor n}{\beta-\alpha}\,\frac{\overline{\beta-x}|^{n+1}-\overline{\alpha-x}|^{n+1}}{(x^2-x+1)^{n+1}}.$$

Now $\alpha - x = \cos\frac{1}{3}\pi - x + \sqrt{(-1)}\sin\frac{1}{3}\pi = \rho\{\cos\phi + \sqrt{(-1)}\sin\phi\}$,

if $\qquad \tan\phi = \dfrac{\sqrt{(3)}}{1-2x}$, and $\rho = \sqrt{(1 - x + x^2)}$,

$\beta - x = \cos\frac{1}{3}\pi - x - \sqrt{(-1)}\sin\frac{1}{3}\pi = \rho\{\cos\phi - \sqrt{(-1)}\sin\phi\}$,

$$\text{or} \quad \frac{d^ny}{dx^n} = \frac{\lfloor n}{2\sqrt{(-1)}\,\rho\sin\phi} \cdot \frac{2\sqrt{(-1)}\,\rho^{n+1}\sin(n+1)\phi}{(x^2 - x + 1)^{n+1}}$$

$$= \frac{\sin(n+1)\phi}{\sin\phi} \cdot \frac{\lfloor n}{\rho^{n+2}},$$

where $\tan\phi$ as above $= \dfrac{\sqrt{(3)}}{1-2x}$, or since $\sin\phi = \dfrac{\sqrt{(3)}}{2\sqrt{(1-x+x^2)}}$,

we may write the result $\lfloor n \dfrac{\sin(n+1)\phi.(\sin\phi)^{n+1}}{(\sin\frac{1}{3}\pi)^{n+2}}$.

This method will apply to all such cases of unreal factors.

The n^{th} differential coefficient of $\tan^{-1}x$ is most conveniently expressed in terms of $\frac{1}{2}\pi - \tan^{-1}x$ which call θ, then $x = \cot\theta$,

$$\text{and} \qquad \frac{dy}{dx} = \frac{1}{1+x^2} = \sin^2\theta;$$

$$\text{therefore} \quad \frac{d^2y}{dx^2} = 2\sin\theta\cos\theta\frac{d\theta}{dx} = -\sin 2\theta . \sin^2\theta;$$

$$\text{because} \qquad \frac{dx}{d\theta} = -\frac{1}{\sin^2\theta};$$

$$\text{therefore} \quad \frac{d^3y}{dx^3} = -\frac{d\theta}{dx}\{2\cos 2\theta\sin^2\theta + 2\sin 2\theta\sin\theta\cos\theta\}$$

$$= \sin^2\theta . 2\sin\theta(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta\}$$

$$= 2\sin^3\theta\sin 3\theta.$$

We observe the law so far to be

$$\frac{d^ny}{dx^n} = (-1)^{n-1}\lfloor n-1 \,\sin n\theta\sin^n\theta,$$

and assuming this to be true for any particular value of n, we at once see that it is true for $n+1$; therefore, &c., the ordinary course of the proof by "Math. Induction."

This may also be proved by the use of De Moivre's Theorem, for

$$\frac{dy}{dx} = \left\{\frac{1}{\sqrt{(-1)} - x} + \frac{1}{\sqrt{(-1)} + x}\right\} \frac{\sqrt{(-1)}}{2};$$

therefore

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{\sqrt{(-1)}}{2} \lfloor n \left[\frac{1}{\{\sqrt{(-1)} - x\}^{n+1}} + \frac{(-1)^n}{\{\sqrt{(-1)} + x\}^{n+1}}\right]$$

$$= \frac{\sqrt{(-1)}}{2} \lfloor n \frac{\{\sqrt{(-1)} + x\}^{n+1} - \{x - \sqrt{(-1)}\}^n}{(-1)^{n+1} (1 + x^2)^{n+1}},$$

and if $x = \cot \theta$,

$$\{x \pm \sqrt{(-1)}\}^{n+1} = \{\cos \theta \pm \sqrt{(-1)} \sin \theta\}^{n+1} \div (\sin \theta)^{n+1}$$

$$= \{\cos (n+1) \theta \pm \sqrt{(-1)} \sin (n+1) \theta)\} \div (\sin \theta)^{n+1};$$

therefore $$\frac{d^{n+1}y}{dx^{n+1}} = -\lfloor n \frac{\sin (n+1) \theta}{(\sin \theta)^{n+1}} \frac{(\sin^2 \theta)^{n+1}}{(-1)^{n+1}}$$

$$= (-1)^n \lfloor n \sin (n+1) \theta \sin^{n+1} \theta.$$

To differentiate $x^n \log x$, take

$$y = x^n \log x, \quad \frac{dy}{dx} = nx^{n-1} \log x + x^n . \frac{1}{x},$$

whence $$x \frac{dy}{dx} = ny + x^n;$$

therefore

$$x \frac{d^{r+1}y}{dx^{r+1}} + r \frac{d^ry}{dx^r} = n \frac{d^ry}{dx^r} + n(n-1)\ldots(n-r+1) x^{n-r};$$

therefore

$$\frac{d^{r+1}y}{dx^{r+1}} = (n-r) \frac{1}{x} \frac{d^ry}{dx^r} + n(n-1)\ldots(n-r+1) x^{n-r-1} \ldots (A).$$

Now $$\frac{dy}{dx} = nx^{n-1} \left\{\log x + \frac{1}{n}\right\},$$

$$\frac{d^2y}{dx^2} = n(n-1) x^{n-2} \left\{\log x + \frac{1}{n} + \frac{1}{n-1}\right\},$$

and if we assume

$$\frac{d^r y}{dx^r}=n(n-1)\ldots(n-r+1)x^{n-r}\left\{\log x+\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r+1}\right\},$$

$$\frac{d^{r+1}y}{dx^{r+1}}=n(n-1)\ldots(n-r)x^{n-r-1}\left\{\log x+\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r+1}\right\}$$
$$+n(n-1)\ldots(n-r+1)x^{n-r+1}$$
$$=\ldots\ldots\ldots\ldots\ldots\ldots\left\{\log x+\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r}\right\},$$

the same rule.

If we put $r=n$ in (A) we get at once $\dfrac{d^{n+1}y}{dx^{n+1}}=\dfrac{\lfloor n}{x}$; whence all subsequent ones are known. For purposes explained in the next section, it is often convenient to obtain a *rational* equation connecting two or more differential coefficients of any function. Suppose, for instance, $y=\sin^{-1}x$, therefore

$$\sqrt{(1-x^2)}\frac{dy}{dx}=1,\quad \sqrt{(1-x^2)}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{(1-x^2)}}\frac{dy}{dx}=0,$$

or
$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0,$$

the required equation. Differentiating this n times, we have

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}+n(-2x)\frac{d^{n+1}y}{dx^{n+1}}+n\frac{(n-1)}{2}(-2)\frac{d^n y}{dx^n}$$
$$-x\frac{d^{n+1}y}{dx^{n+1}}-n(1)\frac{d^n y}{dx^n}=0$$

or
$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}-n^2\frac{d^n y}{dx^n}=0.$$

So if
$$y=(\sin^{-1}x)^2,\quad \sqrt{(1-x^2)}\frac{dy}{dx}=2\sin^{-1}x,$$

$$\sqrt{(1-x^2)}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{(1-x^2)}}\frac{dy}{dx}=\frac{2}{\sqrt{(1-x^2)}},$$

or $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$, which of course leads to the same

equation as before, except that in the former case the equation is true when $n=0$, which is not so here.

The two functions $\sin(m \sin^{-1}x)$, and $\cos(m \sin^{-1}x)$, each satisfy the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0,$$

and therefore also the equation

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0.$$

The function $\varepsilon^{a\sin^{-1}x}$ satisfies the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$

(It will be observed that this may be deduced from the last by putting $a\sqrt{(-1)}$ for m, and therefore the resulting equation connecting the higher differential coefficients is

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}} - (a^2+n^2)\frac{d^ny}{dx^n} = 0).$$

So in general functions of any form may be eliminated by differentiation, and a relation found connecting their differential coefficients. Such a relation is called a differential equation. The principal use made of them in the differential calculus is to find the law connecting the successive coefficients of different powers of x when the function is expanded into a series of integral powers of x. Thus, if we had not the Binomial Theorem, and found the differential coefficient of $(1+x)^n$ to be $n(1+x)^{n-1}$ otherwise (which is not difficult), we should have if $y=(1+x)^n$, the differential equation $(1+x)\dfrac{dy}{dx}=ny$, and assuming y to be

$$= A_0 + A_1x + A_2\frac{x^2}{\underline{|2}} + \ldots + A_r\frac{x^r}{\underline{|r}} + \ldots,$$

then $$(1+x)\left(A_1 + A_2x + \ldots + A_r\frac{x^{r-1}}{\underline{|r-1}} + A_{r+1}\frac{x^r}{\underline{|r}} + \ldots\right)$$

$$= n\left(A_0 + A_1x + \ldots + A_r\frac{x^r}{\underline{|r}} + \ldots\right),$$

or equating coefficients of x^r,

$$\frac{A_r}{(r-1)!} + \frac{A_{r+1}}{r!} = n\frac{A_r}{r!}; \text{ therefore } A_{r+1} = (n-r)A_r;$$

therefore $A_1 = nA_0$, $A_2 = n(n-1)A_0$, &c.,

and therefore

$$y = A_0\left\{1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\right\},$$

which would satisfy the differential equation whatever A_0 be; but since in this case $y=1$ when $x=0$, we have $A_0 = 1$, and obtain the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots .$$

The fact observed here that the solution of this differential equation involves one arbitrary constant and no more is true for all differential equations of what is called the first order, *i.e.* between $\frac{dy}{dx}$, y, x. So the general solution of one of the second degree will involve two arbitrary constants and so on, it being obvious that in the case of an equation, say of the second degree, we may, for any proposed value of x, give y and $\frac{dy}{dx}$ what values we choose, but when these values are chosen $\frac{d^2y}{dx^2}$ and therefore all subsequent differential coefficients are determined by the equation. Thus, in the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0,$$

suppose we choose that when $x=0$, $y=a$, and $\frac{dy}{dx} = b$, and assume accordingly

$$y = a + bx + a_2\frac{x^2}{2!} + \dots + a_r\frac{x^r}{r!} + \dots;$$

therefore

$$(1-x^2)\left(a_2+a_3x+\ldots+a_r\frac{x^{r-2}}{\lfloor r-2}+a_{r+1}\frac{x^{r-1}}{\lfloor r-1}+a_{r+2}\frac{x^r}{\lfloor r}+\ldots\right)$$
$$-x\left(b+a_2x+\ldots+a_r\frac{x^{r-1}}{\lfloor r-1}+\ldots\right)=0,$$

whence, equating coefficients of x^r,

$$\frac{a_{r+2}}{\lfloor r}-\frac{a_r}{\lfloor r-2}-\frac{a_r}{\lfloor r-1}=0,$$

or $$a_{r+2}=a_r\{r(r-1)+r\}=r^2a_r;$$

therefore $a_2=0$; therefore $a_4=0$, $a_6=0$, &c.,

$$a_3=1^2b,\quad a_5=3^2.1^2.b,\quad a_7=5^2.3^2.1^2.b,\ \&c.,$$

and therefore

$$y=a+b\left\{x+1^2.\frac{x^3}{\lfloor 3}+1^2.3^2.\frac{x^5}{\lfloor 5}+1^2.3^2.5^2.\frac{x^7}{\lfloor 7}+\ldots\right\}$$
$$=a+b\left\{x+\frac{1}{2}\,\frac{x^3}{3}+\frac{1.3}{2.4}\,\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\,\frac{x^7}{7}+\ldots\right\}.$$

This is in fact $y=a+b\sin^{-1}x$, which satisfies the equation equally well and is the *general* solution.

DIFFERENTIAL CALCULUS. II.

1. Prove that

$$\frac{dy}{dx}\,\frac{dx}{dy}=1,\quad \frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx}=\frac{dy}{du}\,\frac{dy}{dz}\,\frac{dz}{dx},$$

z being a function of x, and u of z. Find the differential coefficients of

$$\sin^{-1}\left(\frac{x}{a}\right),\quad \cos^{-1}\left(\frac{x}{a}\right),\quad \tan^{-1}\left(\frac{x}{a}\right),\quad \text{vers}^{-1}\left(\frac{x}{a}\right)$$

$$\cos^{-1}\left(\frac{a^2-x^2}{a^2+x^2}\right),\quad \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\quad \log\sin x,\ \log\cos x.$$

$$\left\{ Ans. \quad \frac{1}{\sqrt{(a^2-x^2)}}, \quad \frac{-1}{\sqrt{(a^2-x^2)}}, \quad \frac{a}{a^2+x^2}, \quad \frac{1}{\sqrt{(2ax-x^2)}}, \right.$$

$$\left. \frac{2a}{a^2+x^2}, \quad \frac{3}{1+x^2}, \quad \cot x, \quad -\tan x \right\}.$$

2. From the equality

$$\sin x \sin(\alpha+x) \sin(2\alpha+x) \ldots \sin\{(n-1)\alpha+x\} \equiv 2^{1-n} \sin nx,$$

where $$n\alpha = \pi,$$

find, by taking the logarithmic differential coefficient of both members, the sums of the series

(1) $\cot x + \cot(\alpha+x) + \cot(2\alpha+x) \ldots + \cot\{(n-1)\alpha+x\}$,

(2) $\cot^2 x + \cot^2(\alpha+x) + \cot^2(2\alpha+x) + \ldots + \cot^2\{(n-1)\alpha+x\}$.

3. Having given that

$$\cos x \cos\frac{x}{2} \cos\frac{x}{2^2} \ldots \cos\frac{x}{2^{n-1}} \ldots \text{to } \infty = \frac{\sin 2x}{2x},$$

prove that

(1) $$\tan x + \tfrac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots$$
$$+ \frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}} + \ldots \text{to } \infty = \frac{1}{x} - 2\cot 2x,$$

(2) $$\tan^2 x + \frac{1}{2^2}\tan^2\frac{x}{2^2} + \frac{1}{2^4}\tan^2\frac{x}{2^2} + \ldots$$
$$+ \frac{1}{4^{n-1}}\tan^2\frac{x}{2^{n-1}} + \ldots \text{to } \infty = \tfrac{8}{3} - \frac{1}{x^2} + 4\cot^2 2x.$$

4. Find the differential coefficients of

$$\frac{x}{\sqrt{(a^2-x^2)}}, \quad (2ax+x^2)^{\frac{3}{2}}, \quad \frac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}},$$

$$a \operatorname{vers}^{-1}\left(\frac{x}{a}\right) + \sqrt{(2ax-x^2)}, \quad \tan^{-1}\left(\frac{x}{a}\right) + \log\sqrt{\left(\frac{a-x}{a+x}\right)},$$

$$\frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan\frac{x}{2}\right\},$$

and $$\frac{1}{\sqrt{(b^2-a^2)}} \log\left\{\frac{\sqrt{(b+a)} + \sqrt{(b-a)}\tan\frac{1}{2}x}{\sqrt{(b+a)} - \sqrt{(b-a)}\tan\frac{1}{2}x}\right\}.$$

$$\left\{ Ans. \quad \frac{a^2}{(a^2-x^2)^{\frac{3}{2}}},\ (a-x)\sqrt{(2ax-x^2)},\ \frac{4}{(\varepsilon^x+\varepsilon^{-x})^2},\ \sqrt{\left(\frac{2a-x}{x}\right)},\ \frac{2ax^2}{x^4-a^4},\ \frac{1}{a+b\cos x} \right\}.$$

5. Explain the notation $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots$, and find the n^{th} differential coefficients of $\sin x$, $\cos x$, $\log x$, and $\sin^3 x$. If $y = \cot^{-1}(x)$, prove that

$$\frac{d^n y}{dx^n} = (-1)^n \underline{|n-1} \sin^n y \sin ny.$$

If $y = x^{n-1}\log x$, $\frac{d^{n-1}y}{dx^{n-1}} = \underline{|n-1}\left\{\log x + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}\right\}$.

6. Express $\frac{d^n(yz)}{dx^n}$ in terms of y, z, and their several differential coefficients. Find the n^{th} differential coefficients of $x^3 \sin x$, $(1-x^2)\cot^{-1}(x)$; and prove that the n^{th} differential coefficient of $\varepsilon^{ax}\cos bx$ is

$$(a^2+b^2)^{\frac{n}{2}}\varepsilon^{ax}\cos\left\{bx + n\tan^{-1}\left(\frac{b}{a}\right)\right\}.$$

7. If $y = A\{x+\sqrt{(x^2+a^2)}\}^n + B\{x+\sqrt{(x^2+a^2)}\}^{-n}$,

then will $$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2 y = 0,$$

and $$(x^2+a^2)\frac{d^{r+2}y}{dx^{r+2}} + (2r+1)x\frac{d^{r+1}y}{dx^{r+1}} + (r^2-n^2)\frac{d^r y}{dx^r} = 0.$$

DIFFERENTIAL CALCULUS.
EXPANSIONS OF FUNCTIONS IN SERIES.

Taylor's Theorem.

If we assume that a function $\phi(x+h)$ can be expanded in a series of ascending integral powers of h, and assume that expansion to be $A_0 + A_1 h + A_2\frac{h^2}{\underline{|2}} + \dots + A_n\frac{h^n}{\underline{|n}} + \dots$, where $A_0, A_1 \dots A_n$ do not involve h at all, *i.e.* are func-

tions of x only; then differentiating both sides of this equation (1) with respect to h only, not assuming x to vary (2) with respect to x only, we shall have

$$(1)\quad \phi'(x+h) = A_1 + A_2 h + A_3 \frac{h^2}{\lfloor\underline{2}} + \ldots + A_n \frac{h^{n-1}}{\lfloor\underline{n-1}} + \ldots,$$

$$(2)\quad \phi'(x+h) = \frac{dA_0}{dx} + h\frac{dA_1}{dx} + \ldots + \frac{h^{n-1}}{\lfloor\underline{n-1}}\frac{dA_{n-1}}{dx} + \ldots.$$

Since the differential coefficient of $\phi(z)$ with respect to h is $\phi'(z)\frac{dz}{dh}$, and with respect to x is $\phi'(z)\frac{dz}{dx}$, and when $z = x + h$, $\frac{dz}{dx} = 1$, and $\frac{dz}{dh} = 1$.

From these two expansions of the same function we get

$$A_1 = \frac{dA_0}{dx},\quad A_2 = \frac{dA_1}{dx} = \frac{d^2A_0}{dx^2}\ \ldots,$$

$$A_n = \frac{dA_{n-1}}{dx} = \frac{d^nA_0}{dx^n}.$$

Also putting $h = 0$ in the original expansion, we get $A_0 = \phi(x)$, or the expansion, if such a one can be effected at all, is

$$\phi(x+h) = \phi(x) + h\phi'(x) + \frac{h^2}{\lfloor\underline{2}}\phi''(x) + \ldots + \frac{h^n}{\lfloor\underline{n}}\phi''(x) + \ldots.$$

No information is given by this method as to when this series is divergent, and when convergent, and arithmetically true, and so long as no satisfactory interpretation is given to divergent series, we cannot assert the equality of $\phi(x+h)$ to the series which professes to be its expansion in powers of h, until we have ascertained that the series is convergent, and tends to a finite limit whatever the number of terms taken. Thus if we expand $\frac{1}{x-h}$ by this series, we get of course the expansion

$$\frac{1}{x} + \frac{h}{x^2} + \frac{h^2}{x^3} + \ldots + \frac{h^n}{x^{n+1}} + \ldots.$$

In this particular case, we know from ordinary algebra, the remainder of the series after n terms, viz. $\frac{h^n}{x^n(x-h)}$, and see that the series is convergent only when $\frac{h}{x}$ is numerically < 1, so that the remainder may be made as small as we please by taking a sufficient number of terms. It is then advisable in general to determine *limits* for the remainder after n terms of this expansion, and slightly altering the notation, we assume

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{\lfloor 2}\phi''(a) + \ldots$$
$$+ \frac{h^{n-1}}{\lfloor n-1}\phi^{n-1}(a) + \frac{h^n}{\lfloor n}R \ldots\ldots\ldots\ldots\ldots (A),$$

and seek to determine the form of R in some way. Now denote the following function of x

$$\phi(a+x) - \phi(a) - x\phi'(a) - \frac{x^2}{\lfloor 2}\phi''(a) - \ldots - \frac{x^n}{\lfloor n}R,$$

by $F(x)$, then by equation (A), we have $F(h) = 0$. Also $F(0) = \phi(a) - \phi(a) = 0$. Hence, $F(x)$ vanishes for the two particular values of x, 0, and h. But if a function of x vanish for two particular values of x and do not become infinite between those limits, then since it cannot be always increasing or always decreasing, it must at some point change from increasing to decreasing or the reverse, *i.e.* its differential coefficient must change sign, which not being infinite, it cannot do without passing through the value 0. Hence $F'(x)$ vanishes for some value of x between 0 and h (x_1 suppose). But

$$F'(x) = \phi'(a+x) - \phi'(a) - x\phi''(a) - \ldots - \frac{x^{n-1}}{\lfloor n-1}R,$$

and therefore vanishes when $x = 0$. Hence, the same argument applies to $F'(x)$, and its differential coefficient or $F''(x)$ must vanish for some value of x between 0

and x_1; (x_2 suppose), which is *a fortiori* between 0 and h. But

$$F''(x) = \phi''(a+x) - \phi''(a) - x\phi'''(a) - \ldots - \frac{x^{n-2}}{\lfloor n-2} R,$$

which also vanishes when $x=0$; and the same again applies, and so on, until we come to the result that $F^n(x)$, or $\phi^n(a+x) - R$ must vanish for some value of x between 0 and h. (This does not vanish when $x=0$, so far as we know; and, therefore, the argument does no longer apply). Thus $R = \phi^n(a+\theta h)$ where θ is some positive proper fraction, so that we have

$$\phi(a+h) = \phi(a) + h\,\phi'(a) + \frac{h^2}{\lfloor 2}\phi''(a) + \ldots + \frac{h^n}{\lfloor n}\phi^n(a+\theta h),$$

θ being some positive proper fraction, provided that as was assumed in the proof, $\phi(x)$, and all its differential coefficients are finite (*i.e.* not infinitely great, they may vanish) and continuous between the limits a and $a+h$. Hence, if we find the least and greatest values which $\phi^n(x)$ can have between these limits we shall have certain limits between which R must lie.

Thus suppose $\phi(x) = \sqrt{(x)}$, $a=1$, $h = \frac{1}{100} = {\cdot}01$, $n=3$, and we have

$$\sqrt{\left(\frac{101}{100}\right)} = \sqrt{(1)} + \frac{1}{100}\cdot\frac{1}{2\sqrt{(+1)}} + \frac{1}{2}\left(\frac{1}{100}\right)^2\cdot\frac{-1}{4\times 1^{\frac{3}{2}}}$$
$$+\frac{1}{\lfloor 3}\cdot\left(\frac{1}{100}\right)^3\cdot\frac{3}{8\left(1+\frac{\theta}{100}\right)^{\frac{5}{2}}}$$
$$= 1 + \frac{1}{200} - \frac{1}{80000} + \frac{1}{16000000}\,\frac{1}{\left(1+\frac{\theta}{100}\right)^{\frac{5}{2}}}.$$

Hence the sum of the first three terms will differ from the truth by $< \frac{1}{16000000}$, or to eight decimal places

$$\sqrt{\left(\frac{101}{100}\right)} = 1 + {\cdot}005 - {\cdot}0000125, \text{ or } \sqrt{(101)} = 10{\cdot}049875.$$

θ will in general be a very complicated function both of a and h, and it would be a very difficult matter to determine it; but the important property of it is that it must always lie between 0 and 1.

If $\phi(x)$ be a rational algebraical function of the $n+1^{\text{th}}$ degree, $\theta = \dfrac{1}{n+1}$, which appears to be the only case in which it is constant. If $\phi(x)$ be ε^x, θ is a function of h only, given by the equation

$$\varepsilon^{a+h} = \varepsilon^a + h\varepsilon^a + \frac{h^2}{\lfloor 2}\varepsilon^a + \ldots + \frac{h^n}{\lfloor n}\varepsilon^{a+\theta h},$$

or $$\theta = \frac{1}{h}\log\left\{\frac{\lfloor n}{h^n}\left(\varepsilon^h - 1 - h - \frac{h^2}{\lfloor 2} - \ldots - \frac{h^{n-1}}{\lfloor n-1}\right)\right\},$$

a function of h only.

If $\phi(x)$ be x^5 and $n = 3$, we have

$$(a+h)^5 = a^5 + h\,5a^4 + \frac{h^2}{2}\,20a^3 + \frac{h^3}{6}\,60\,(a+6h)^2$$

$$= a^5 + 5a^4h + 10a^3h^2 + 10a^2h^3 + 5ah^4 + h^5,$$

so that $$(a+\theta h)^2 = a^2 + \tfrac{1}{2}ah + \tfrac{1}{10}h^2,$$

or $\theta = -\dfrac{a}{h} + \sqrt{\left(\dfrac{a^2}{h^2} + \dfrac{1}{2}\dfrac{a}{h} + \dfrac{1}{10}\right)}$; taking the $+$ sign since θ is positive

$$= \left(\frac{1}{2}\frac{a}{h} + \frac{1}{10}\right) \div \frac{a}{h} + \sqrt{\left(\frac{a^2}{h^2} + \frac{a}{2h} + \frac{1}{10}\right)}.$$

The greatest and least possible values of θ are found by putting $h = \infty$, $h = 0$, and are $\dfrac{1}{\sqrt{(10)}}$ and $\dfrac{1}{4}$. Thus

$$\theta > \frac{1}{4} \text{ if } \left(\frac{a}{h} + \frac{1}{4}\right)^2 < \frac{a^2}{h^2} + \frac{1}{2}\frac{a}{h} + \frac{1}{10},$$

i.e. if $$\frac{1}{16} < \frac{1}{10};$$

so also $\theta < \dfrac{1}{\sqrt{(10)}}$ if $\left(\dfrac{a}{h} + \dfrac{1}{\sqrt{10}}\right)^2 > \dfrac{a^2}{h^2} + \dfrac{1}{2}\dfrac{a}{h} + \dfrac{1}{10}$,

or if $$\frac{1}{\sqrt{(10)}} > \frac{1}{4}.$$

(We have assumed $\frac{a}{h}$ to be positive, if it be negative we ought to take the other sign in the ambiguity).

Putting $a=0$ in the general formula, we have

$$f(h) = f(0) + hf'(0) + \frac{h^2}{\underline{|2}} f''(0) + \dots$$
$$+ \frac{h^{n-1}}{\underline{|n-1}} f^{n-1}(0) + \frac{h^n}{\underline{|n}} f^n(\theta h),$$

or, replacing h by x,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \dots + \frac{x^n}{\underline{|n}} f^n(\theta x),$$

θ denoting some positive proper fraction; and all the functions $f(x)$, $f'(x) \dots f^n(x)$ being finite between the limits 0 and x. Thus, if $f(x)$ be $\log x$, we cannot obtain an expansion since $f(x), f'(x) \dots$ all become infinite when $x=0$. So also if $f(x)$ be $\varepsilon^{-\frac{1}{x}}$, $\varepsilon^{-\frac{1}{x^2}} \dots$.

If $f(x)$ or y be $\sin^{-1}(x)$, we have seen that

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} - n^2\frac{d^ny}{dx^n} = 0,$$

hence, putting $x=0$ in this equation,

$$f^{n+2}(0) = n^2 f^n(0),$$

from which we can find $f''(0)$, $f'''(0) \dots$ after finding $f(0)$ and $f'(0)$. But $f(0)=0$, $f'(0)=1$, whence

$$f''(0)=0,\ f^{iv}(0)=0,\ f^{vi}(0)=0,\ \dots,$$

and $$f'''(0)=1,\ f^{v}(0)=3^2,\ f^{vii}(0)=3^2.5^2,\ \dots,$$

or $$\sin^{-1}(x) = x + \frac{x^3}{\underline{|3}} + 3^2\frac{x^5}{\underline{|5}} + 3^2.5^2\frac{x^7}{\underline{|7}} + \dots$$

$$\equiv x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{7.4.6}\frac{x^7}{7} + \dots \text{ to } \infty \dots (A),$$

a series which is necessarily convergent since x is <1 for real values of $\sin^{-1}(x)$.

The same differential equation holds for $(\sin^{-1}x)^2$ for all values of n after 1, but in this case

$$f(0)=0,\ f'(0)=0,\ f''(0)=2;$$

therefore $f^{iv}(0)=2^2.2$, $f^{vi}(0)=4^2.2^2.2$, and so on,

while $$f'''(0)=0,\ f^{v}(0)=0,\ \&c.,$$

so that $(\sin^{-1}x)^2=2.\dfrac{x^2}{\lfloor 2}+2.2^2\dfrac{x^4}{\lfloor 4}+2.2^2.4^2\dfrac{x^6}{\lfloor 6}+\ldots,$

or $$\frac{(\sin^{-1}x)^2}{2}=\frac{x^2}{2}+\frac{2}{3}\,\frac{x^4}{4}+\frac{2.4}{3.5}\,\frac{x^6}{6}+\frac{2.4.6}{3.5.7}\,\frac{x^8}{8}+\ldots\ldots\ (B).$$

It is a singular circumstance, that if in (B) every figure whatever (except that in $\sin^{-1}x$) be diminished by 1, we get the series (A).

For the functions $\sin(m\sin^{-1}x)$, $\cos(m\sin^{-1}x)$, we get the equation

$$f^{n+2}(0)=(n^2-m^2)f^n(0),$$

and since $f(0)=0$ and $f'(0)=m$ for the first, and $f(0)=1$, $f'(0)=0$ for the second, we shall have

$\sin(m\sin^{-1}x)$

$$=mx-m(m^2-1^2)\frac{x^3}{\lfloor 3}+m(m^2-1^2)(m^2-3^2)\frac{x^5}{\lfloor 5}-\ldots,$$

$\cos(m\sin^{-1}x)$

$$=1-m^2\frac{x^2}{\lfloor 2}+m^2(m^2-2^2)\frac{x^4}{\lfloor 4}-m^2(m^2-2^2)(m^2-4^2)\frac{x^6}{\lfloor 6}+\ldots,$$

both of which are convergent since x is here also <1. These series are true for all values of m, but since in finding $f(0)$, $f'(0)$, we take $\sin^{-1}(0)$ to be 0, we must always suppose $\sin^{-1}(x)$ to be an acute angle positive or negative, according to the general rule already laid down as a convenient universal rule.

For the expansion of $\varepsilon^{a\sin^{-1}x}$, we have

$$f^{n+2}(0)=(a^2+n^2)f^n(0),$$

also $$f(0)=1,\ f'(0)=a;$$

therefore $$f''(0)=a^2,\ f'''(0)=a(a^2+1^2),$$

and so on, or

$$\varepsilon^{a\sin^{-1}x}=1+\frac{a^2x^2}{\lfloor 2}+a^2(a^2+2^2)\frac{x^4}{\lfloor 4}+a^2(a^2+2^2)(a^2+4^2)\frac{x^6}{\lfloor 6}+\dots$$
$$+ax+a(a^2+1^2)\frac{x^3}{\lfloor 3}+a(a^2+1^2)(a^2+3^2)\frac{x^5}{\lfloor 5}+\dots,$$

which really includes both the series found in the last article, since putting $m\sqrt{(-1)}$ for a, we get

$$\varepsilon^{a\sin^{-1}x}=\cos(m\sin^{-1}x)+\sqrt{(-1)}\sin(m\sin^{-1}x),$$

and equating real parts, and also unreal parts, we obtain both expansions. In this case also $\sin^{-1}(x)$ always denotes an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Again, since

$$\varepsilon^{a\sin^{-1}x}=1+a\sin^{-1}x+\frac{a^2}{\lfloor 2}(\sin^{-1}x)^2+\frac{a^3}{\lfloor 3}(\sin^{-1}x)^3+\dots,$$

we may obtain the series already found for $(\sin^{-1}x)$, $(\sin^{-1}x)^2$ by picking out the coefficients of a, a^2, and similarly we shall get for $(\sin^{-1}x)^3$ and $(\sin^{-1}x)^4$,

$$\frac{(\sin^{-1}x)^3}{\lfloor 6}=1^2\frac{x^3}{\lfloor 3}+(1^2+3^2)\frac{x^5}{\lfloor 5}$$
$$+1^23^25^2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{x^7}{\lfloor 7}+\dots\dots\dots\dots (C),$$

$$\frac{(\sin^{-1}x)^4}{\lfloor 4}=\frac{x^4}{\lfloor 4}+2^2.4^2\left(\frac{1}{2^2}+\frac{1}{4^2}\right)\frac{x^6}{\lfloor 6}$$
$$+2^2.4^2.6^2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}\right)\frac{x^8}{\lfloor 8}+\dots\dots\dots (D),$$

C and D being connected in nearly the same way as the series (A), (B) for $\sin^{-1}x$ and $(\sin^{-1}x)^2$.

These series may be written

$$\frac{(\sin^{-1} x)^3}{\underline{|3}} = \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\left(\frac{1}{1^2} + \frac{1}{3^2}\right)\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right)\frac{x^7}{7} + \ldots,$$

$$\frac{(\sin^{-1} x)^4}{\underline{|4}} = \frac{2}{3}\cdot\frac{1}{2^2}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\left(\frac{1}{2^2} + \frac{1}{4^2}\right)\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\right)\frac{x^8}{8} + \ldots.$$

The series for $\sin x$, $\cos x$, $\tan^{-1} x$ are found in all works on Plane Trigonometry, and can of course readily be obtained by Maclaurin's series; $\tan^{-1} x$ can however be easily found by integration, thus

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots \text{to } \infty \text{ (when } x < 1);$$

therefore $$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots,$$

and since when $x = 0$, $\tan^{-1} x = 0$, $C = 0$; or the series is completely determined. Of course $\tan^{-1}(x)$ must be an angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and since, for the series to be convergent, $x < 1$, $\tan^{-1}(x)$ must lie between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$.

The following functions of x will furnish good examples for the student:

(1) $$\frac{\{x + \sqrt{(1+x^2)}\}^n + \{x + \sqrt{(1+x^2)}\}^{-n}}{2},$$

(2) $$\frac{\sin(n \tan^{-1} x)}{(1+x^2)^{\frac{1}{2}n}},$$ (3) $$\tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right),$$

giving in (3) the remainder after the first n significant terms.

Also $\tan^{-1}(x+h)$ can be expanded in terms of h, since $\left(\frac{d}{dx}\right)^n (\tan^{-1} x)$ has been found.

The geometrical proof of the equation

$$\phi(x+h) = \phi(x) + h\phi'(x + \theta h),$$

is so simple, and this equation is of itself so useful, that we may well insert it. Suppose we measure a distance x (fig. 11) along a fixed straight line Ox, from a fixed point O, and at the end of any distance x erect a perpendicular to Ox, of length $= \phi(x)$, the ends of these straight lines or ordinates will trace out a curve of some form, PQ suppose; let $OM = x$, and, therefore,

$$MP = \phi(x),\quad ON = x + h,\quad NQ = \phi(x+h),\quad PL = MN = h,$$

then
$$LQ = NQ - NP = \phi(x+h) - \phi(x),$$

or
$$\frac{\phi(x+h) - \phi(x)}{h} = \tan QPL.$$

Now whatever the form of the curve PQ, provided it is continuous and does not move to an infinite distance between P and Q, the tangent at some point between P and Q must be parallal to PQ, at R suppose, so that if RT be the tangent at R, $\angle RTx = \angle QPL$, but if X be the value of OU the *abscissa* of R, $\tan RTx = \phi'(X)$ as was proved in the first chapter; and since X lies between x and $x + h$, we may denote it by $x + \theta h$, so that the equation $\tan QPL = \tan RTx$, gives us

$$\frac{\phi(x+h) - \phi(x)}{h} = \phi'(x + \theta h),$$

i.e.
$$\phi(x+h) = \phi(x) + h\phi'(x + \theta h),$$

θ being a positive proper fraction, and $\phi(x)$ a continuous function of x, which does not become infinite between the values x and $x + h$.

This equation alone furnishes proof of the theory of "Proportional Parts" in taking logarithms &c. from tables and indicates the exceptional cases, which are when $\phi'(x)$ is either very large or very small, so that its changes between x and $x + h$ are either themselves very large, or are large compared with the whole difference wanted.

The limiting value of θ in this equation, when h is indefinitely diminished, is always $\frac{1}{2}$. For

$$\phi(x+h) = \phi(x) + h\phi'(x) + \frac{h^2}{2}\phi''(x + nh),$$

n being a positive proper fraction,

$$= \phi(x) + h\phi'(x + \theta h),$$

and $$\phi'(x + \theta h) = \phi'(x) + \theta h \phi''(x + mh),$$

m being a positive fraction $< \theta$; therefore

$$\frac{h^2}{2}\phi''(x + nh) = \theta h^2 \phi''(x + mh),$$

$$\theta = \frac{1}{2}\frac{\phi''(x + nh)}{\phi''(x + mh)} = \frac{1}{2} \text{ when } h = 0.$$

(This indicates that in any curve PQ if the tangent at R be parallel to PQ, then when PQ moves parallel to itself up to R, R ultimately bisects the arc PQ).

Another proof of the theorem on the limits of the remainder after n terms is somewhat shorter than Homersham Cox's, which is the one we have given. This is as follows: Assume

$$\phi(X) = \phi(x) + (X - x)\phi'(x) + \frac{(X - x)^2}{\lfloor 2}\phi''(x) + \ldots$$
$$+ \frac{(X - x)^{n-1}}{\lfloor n - 1}\phi^{n-1}(x) + \frac{(X - x)^n}{\lfloor n} R \ldots\ldots\ldots (A),$$

and let $F(z) = \phi(X) - \phi(z) - (X - z)\phi'(z)$
$$- \frac{(X - z)^2}{\lfloor 2}\phi''(z) - \ldots - \frac{(X - z)^n}{\lfloor n} R,$$

then $F(x) = 0$ by reason of equation (A), and

$$F(X) = \phi(X) - \phi(X) = 0,$$

and since $F(z)$ vanishes for the two values x and X, its differential coefficient must vanish for some intermediate value $\{$say $x + \theta(X - x)\}$, but

$$F'(z) = -\phi'(z) + \phi'(z) - (X - z)\phi''(z) + (X - z)\phi''(z)$$
$$- \frac{(X - z)^2}{\lfloor 2}\phi'''(z) + \ldots + \frac{(X - z)^{n-1}}{\lfloor n - 1}\phi^n(z) - \frac{(X - z)^{n-1}}{\lfloor n - 1} R$$
$$= \frac{(X - z)^{n-1}}{\lfloor n - 1}\{\phi^n(z) - R\},$$

all the other terms cancelling. Hence

$$R = \phi^n(z) = \phi^n\{x + \theta(X - x)\}$$

as before, or writing $X = x + h$, we have

$$\phi(x+h) = \phi(x) + h\phi'(x) + \frac{h^2}{\lfloor 2}\phi''(x) + \ldots + \frac{h^n}{\lfloor n}\phi^n(x+\theta h).$$

If $\phi'(x)$ be a function which either increases with x or decreases as x increases, since

$$\phi(x+h) = \phi(x) + h\phi'(x+\theta h),$$

we have, putting $h = 1$,

$$\phi(x+1) - \phi(x) = \phi'(x+\theta),$$

which, under the circumstances supposed, always lies between $\phi'(x)$ and $\phi'(x+1)$. Thus if $\phi(x)$ be $\sec^{-1}(x)$, and therefore

$$\phi'(x) = \frac{1}{x\sqrt{(x^2-1)}},$$

we have

$$\sec^{-1}(x+1) - \sec^{-1}(x) > \frac{1}{(x+1)\sqrt{(x^2+2x)}} < \frac{1}{x\sqrt{(x^2-1)}},$$

or if we take $x = n, n-1, n-2, \ldots 1$, successively, we have

$$\sec^{-1}(n+1) - \sec^{-1}(n) > \frac{1}{(n+1)\sqrt{(n^2+2n)}} < \frac{1}{n\sqrt{(n^2-1)}},$$

$$\sec^{-1}n - \sec^{-1}(n-1) > \frac{1}{n\sqrt{(n^2-1)}} < \frac{1}{(n-1)\sqrt{(n^2-2n)}},$$

$$\ldots\ldots\ldots > \ldots\ldots\ldots\ldots\ldots < \ldots\ldots\ldots$$

$$\sec^{-1}(3) - \sec^{-1}(2) > \frac{1}{3\sqrt{(8)}} < \frac{1}{2\sqrt{(3)}},$$

$$\sec^{-1}(2) - \sec^{-1}(1) > \frac{1}{2\sqrt{(3)}},$$

the other limit not applying here (though quite true); therefore

$$\sec^{-1}(n) > \frac{1}{2\sqrt{(3)}} + \frac{1}{3\sqrt{(8)}} + \ldots + \frac{1}{n\sqrt{(n^2-1)}},$$

$$\sec^{-1}(n+1) - \sec^{-1}(2) < \frac{1}{2\sqrt{(3)}} + \frac{1}{3\sqrt{(8)}} + \ldots + \frac{1}{n\sqrt{(n^2-1)}},$$

or $\quad \sum_{r=2}^{r=n} \frac{1}{r\sqrt{(r^2-1)}} > \sec^{-1}(n+1) - \frac{1}{3}\pi$ and $< \sec^{-1}(n)$.

So, in general, the sum of the series

$$\phi'(1) + \phi'(2) + \ldots + \phi'(n),$$

lies between the limits

$$\phi(n) - \phi(0), \text{ and } \phi(n+1) - \phi(1).$$

Thus, if $\qquad \phi'(x) = \sec^2\theta x, \; \phi(x) = \frac{\tan x\theta}{\theta},$

and $\sec^2\theta + \sec^2 2\theta + \ldots + \sec^2 n\theta$ lies between $\frac{\tan n\theta}{\theta}$ and $\frac{\tan(n+1)\theta - \tan\theta}{\theta}$, provided that $\sec^2 x\theta$ increase with x throughout; and, therefore, $x\theta < \frac{1}{2}\pi$ always, therefore $(n+1)\theta < \frac{1}{2}\pi$. This method generally gives limits for the sum of any series which are of some value, except when $\phi'(x)$ is tending to ∞ at one end of the series. Thus, if $n\theta$ in the last be near $\frac{1}{2}\pi$, the difference of these limits will be large, and they therefore not of much use.

DIFFERENTIAL CALCULUS. III.

1. If $y(1+x^2)^{\frac{1}{2}n} = \sin(n \tan^{-1} x)$, prove that

$$(1+x^2)\frac{d^2y}{dx^2} + 2(n+1)x\frac{dy}{dx} + n(n+1)y = 0;$$

and thence that

$$(1+x^2)\frac{d^{r+1}y}{dx^{r+1}} + 2(n+r)\frac{d^ry}{dx^r} + (n+r)(n+r-1)\frac{d^{r-1}y}{dx^{r-1}} = 0.$$

2. Prove that if $f(x+h)$ can be expanded in a series of ascending integral powers of h, that expansion will be

$$f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \ldots + \frac{h^n}{\lfloor n}f^n(x) + \ldots.$$

Expand $\sin(x+h)$ in this manner, and deduce the expansions of $\sin h$ and $\cos h$.

3. If $f(x)$ and all its differential coefficients be finite and continuous, then will

$$f(x+h)=f(x)+hf'(x)+\ldots+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(x)+\frac{h^n}{\lfloor n}f^n(x+\theta h),$$

where θ is a positive proper fraction.

If $f(x)$ be x^4 and $n=3$, $\theta=\frac{1}{4}$; and if $f(x)$ be x^5 and $n=3$, $\theta>\frac{1}{4}$ and $<\frac{1}{\sqrt{(10)}}$. Prove that the limiting value of θ when h is indefinitely diminished is $\frac{1}{n+1}$.

4. Deduce the series for the expansion of $f(x)$ in a series of ascending powers of x. Prove that

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$$
$$+\frac{x^{2n-1}}{2n-1}+\frac{x^{2n}}{4n}\left\{\frac{1}{(1-\theta x)^{2n}}-\frac{1}{(1+\theta x)^{2n}}\right\},\quad \theta>0<1.$$

5. Give a geometrical proof of the equation

$$f(x+h)=f(x)+hf'(x+\theta h);$$

and if $f(x)$ be $a+bx+cx^2$, prove that $\theta=\frac{1}{2}$.

6. Expand $\frac{\{x+\sqrt{(x^2+1)}\}^n+\{x+\sqrt{(x^2+1)}\}^{-n}}{2}$ in a series of ascending powers of x.

The answer is

$$1+n^2\frac{x^2}{\lfloor 2}+n^2(n^2-2^2)\frac{x^4}{\lfloor 4}+n^2(n^2-2^2)(n^2-4^2)\frac{x^6}{\lfloor 6}+\ldots.$$

7. Expand $\sin(m\sin^{-1}x)$, and $\cos(m\sin^{-1}x)$, in a series of ascending powers of x.

8. Expand $\cot^{-1}(x+h)$ in a series of ascending powers of h.

The expansion is

$$\theta-h\sin\theta.\sin\theta+\frac{h^2}{\lfloor 2}\sin^2\theta\sin 2\theta-\frac{h^3}{\lfloor 3}\sin^3\theta\sin 3\theta+\ldots,$$

where $\theta=\cot^{-1}(x)$, (of course being between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$).

Hence prove that

$$\tfrac{1}{2}\pi - \theta = \sin\theta\cos\theta + \frac{\sin 2\theta \cos^2\theta}{2} + \frac{\sin 3\theta \cos^3\theta}{3} + \dots .$$

9. If $f'(x)$ continually increase as x increases, or continually diminish, prove that $f(n+1) - f(n)$ lies between $f'(n)$ and $f'(n+1)$.

Hence prove that

(1) $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \log(n+1)$ and $< 1 + \log n$,

(2) $\sec^2\theta + \sec^2 2\theta + \dots + \sec^2 n\theta > \dfrac{\tan n\theta}{\theta}$ and $< \dfrac{\tan(n+1)\theta}{\theta} - 1$,

provided that $(n+1)\theta < \frac{1}{2}\pi$.

Differential Calculus.

Indeterminate Forms.

If $f(x)$, $\phi(x)$ be two functions of x which both vanish when $x = a$, the fraction $f(x) \div \phi(x)$ becomes unmeaning, but as x approaches a, $\dfrac{f(x)}{\phi(x)}$ will generally tend to some limit from which it may be made to differ by less than any assignable quantity before $x = a$. Thus, $1 - x$, and $1 - x^2$ both vanish when $x = 1$; but

$$\frac{1-x}{1-x^2} = \frac{1}{1+x}, \text{ and therefore } \frac{1-x}{1-x^2} - \frac{1}{2} = \frac{1}{2}\cdot\frac{1-x}{1+x},$$

which as x approaches 1 diminishes, and may be made less than any assignable quantity before $x = 1$. Hence the limit of the fraction $\dfrac{1-x}{1-x^2}$, as x approaches 1, is $\frac{1}{2}$. This result is generally written $\left(\dfrac{1-x}{1-x^2}\right)_{x=1} = \dfrac{1}{2}$, but the correct meaning should be carefully borne in mind.

Now, in general, if $f(a) = 0$, we have, putting $x = a + z$,

$$f(x) = f(a+z) = f(a) + zf'(a + \theta z) = zf'(a + \theta z),$$

and so $$\phi(x) = z\phi'(a + \theta z);$$

therefore $$\frac{f(x)}{\phi(x)} = \frac{f'(a+\theta z)}{\phi'(a+\theta z)};$$

and therefore limit of

$$\left\{\frac{f(x)}{\phi(x)}\right\}_{x=a} = \frac{f'(a)}{\phi'(a)},$$

or, more correctly, $\text{limit}_{x=a}\,\dfrac{f(x)}{\phi(x)} = \text{limit}_{x=a}\,\dfrac{f'(x)}{\phi'(x)}$.

This equation is the fundamental one for determining the limits of functions which assume an unmeaning form. If $f'(a)$ and $\phi'(a)$ do not *both* vanish, the limit is $f'(a) \div \phi'(a)$, but if they do, we must continue the process, and we shall have

$$\left\{\frac{f(x)}{\phi(x)}\right\}_{x=a} = \left\{\frac{f'(x)}{\phi(x)}\right\}_{x=a} = \left\{\frac{f''(x)}{\phi''(x)}\right\}_{x=a},$$

and so on so long as *both* numerator and denominator vanish; the limit being attained when one or both are finite. Thus, if $f(a), f'(a) \ldots f^{n-1}(a)$; $\phi(a), \phi'(a) \ldots \phi^{n-1}(a)$ all $= 0$, and $f^n(a)$, $\phi^n(a)$ be one or both finite, the limit of $\dfrac{f(x)}{\phi(x)}$ is $\dfrac{f^n(a)}{\phi^n(a)}$.

(1) All other unmeaning forms can be reduced to this, thus, if $f(a) = \infty$ and $\phi(a) = \infty$,

$$\frac{f(x)}{\phi(x)} = \frac{1}{\phi(x)} \div \frac{1}{f(x)}, \text{ which is of the form } \frac{0}{0};$$

and, therefore,

$$\left\{\frac{f(x)}{\phi(x)}\right\}_{x=a} = \left[\frac{\phi'(x)}{\{\phi(x)\}^2} \div \frac{f'(x)}{\{f(x)\}^2}\right]_{x=a} = \left\{\left(\frac{f(x)}{\phi(x)}\right)^2 \cdot \frac{\phi'(x)}{f'(x)}\right\}_{x=a}.$$

Hence, if the limit be *finite*, we shall have, dividing by $\left\{\dfrac{f(x)}{\phi(x)}\right\}^2$, limit $\left\{\dfrac{\phi(x)}{f(x)}\right\}_{x=a} =$ limit $\left\{\dfrac{\phi'(x)}{f'(x)}\right\}_{x=a}$,

so that in this case the same rule applies as for a fraction of the form $\dfrac{0}{0}$.

(2) The form $\infty \times 0$; if

$$f(a) = 0,\ \ \phi(a) = \infty,\ \ \phi(x) \cdot f(x) = f(x) \div \frac{1}{\phi(x)},$$

which is of the form $\frac{0}{0}$, and to be evaluated the same way. If more convenient, we may reduce to the form $\frac{\infty}{\infty}$, which has been shewn to be subject to the same rule.

(3) The form $\infty - \infty$; if $f(a) = \infty$ and $\phi(a) = \infty$, then

$$f(x) - \phi(x) = f(x)\left\{1 - \frac{\phi(x)}{f(x)}\right\};$$

now if the limit of $\frac{\phi(x)}{f(x)}$ be 1, this is of the form $\infty \times 0$, and may, therefore, be treated as in (2), and if the limit of $\frac{\phi(x)}{f(x)}$ be different from 1, the limit of $f(x) - \phi(x)$ is $\pm\infty$. Thus

$$\sec x - \tan x \equiv \sec x\left(1 - \frac{\tan x}{\sec x}\right) = \sec x\,(1 - \sin x) = \frac{1 - \sin x}{\cos x},$$

and when $x = \frac{1}{2}\pi$, the limit of this is $\left(\frac{-\cos x}{-\sin x}\right)_{x=\frac{1}{2}\pi}$ or 0.

So also $\frac{1}{x^2} - \cot^2 x = \frac{1}{x^2}\left(1 - \frac{x^2}{\tan^2 x}\right)$, which since we know that $\left(\frac{x}{\tan x}\right)_{x=0}$ is 1, takes the form $\infty \times 0$, and we may write this

$$= \frac{\sin^2 x - x^2\cos^2 x}{x^2\tan^2 x} = \frac{\sin x - x\cos x}{x^3} \cdot \frac{\sin x + x\cos x}{x} \cdot \left(\frac{x}{\tan x}\right)^2,$$

the product of three fractions which all take the form $\frac{0}{0}$ when $x = 0$, but of which we know the limit of two; *i.e.*

$$\left(\frac{x}{\tan x}\right)_{x=0} = 1\,;\ \text{and}\ \left(\frac{\sin x + x\cos x}{x}\right)_{x=0} = \left(\frac{\sin x}{x} + \cos x\right)_{x=0} = 2.$$

The remaining one

$$\left(\frac{\sin x - x\cos x}{x^3}\right)_{x=0} = \left(\frac{\cos x - \cos x + x\sin x}{3x^2}\right)_{x=0} = \frac{1}{3}.$$

Hence $$\left(\frac{1}{x^2} - \cot^2 x\right)_{x=0} = \frac{2}{3}.$$

The three indeterminate forms 1^∞, ∞°, 0° can be reduced to forms already discussed, by taking the logarithm in

each case; for if $y = u^v$, $\log y = v \log u$, or $= \dfrac{\log u}{\frac{1}{v}}$, which becomes either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

The two latter cases are not essentially distinct, one being the reciprocal of the other; and the limit is in nearly all cases 1. For taking $v = f(x)$, $u = \phi(x)$,

$$\log y = f(x) \log \phi(x) = \frac{\log \phi(x)}{\dfrac{1}{f(x)}},$$

and its limit is therefore = the limit of $\dfrac{\phi'(x)}{\phi(x)} \div \dfrac{-f'(x)}{\{f(x)\}^2}$ or $\log y = -$ limit of $\dfrac{\phi'(x)}{f'(x)} \cdot \dfrac{f(x)}{\phi(x)} \cdot f(x)$. Now if $f(a) = 0$ and $\phi(a) = 0$, the limit of $\dfrac{f(x)}{\phi(x)}$ = limit of $\dfrac{f'(x)}{\phi'(x)}$, and therefore the limit of $\dfrac{\phi'(x)}{f'(x)} \cdot \dfrac{f(x)}{\phi(x)}$ is 1 in all cases in which the limit of $\dfrac{f(x)}{\phi(x)}$ is finite; and therefore in all such cases the limit of $\log y$ is 0, or that of y is 1. In fact, it appears that the limit of y is 1 in all cases in which the limit of $\dfrac{\phi'(x)}{f'(x)} \cdot \dfrac{f(x)}{\phi(x)}$ is finite; now since both $f(x)$ and $\phi(x)$ vanish when $x = a$, they can both, generally, be expanded in positive powers of $x - a$, and in such case if m, n be the lowest powers of $x - a$ in $f(x)$, $\phi(x)$ respectively, we shall have

$$f(x) = A(x-a)^m (1+\theta), \quad \phi(x) = B(x-a)^n (1+\theta'),$$

where θ, θ' vanish when $x = a$; therefore

$$f'(x) = mA(x-a)^{m-1}(1+\mu), \quad \phi'(x) = nB(x-a)^{n-1}(1+\mu'),$$

where μ and μ' vanish when $x = a$; therefore

$$\frac{f(x)}{\phi(x)} \cdot \frac{\phi'(x)}{f'(x)} = \frac{n(1+\theta)(1+\mu')}{m(1+\theta')(1+\mu)},$$

and the limit when $x = a$ is $\dfrac{n}{m}$, so that the limit of y is 1.

Hence, the only case in which functions which take the form 0^0 or ∞^0 can have any other limit than 1, is where one or both of the functions vanish when $x = a$, but do not admit of any expansion in positive powers of $x - a$. Such forms are

$$(x-a)^r \log(x-a),\ \varepsilon^{\frac{-1}{x-a}},\ \varepsilon^{\frac{-1}{(x-a)^2}}, \text{\&c.}$$

The forms 0^0, ∞^0, may then usually be interpreted 1, although one may with some trouble invent cases in which their limits differ from 1. Thus, if

$$u = \varepsilon^{-\frac{1}{x^2}}, \text{ and } v = 1 - \cos 2x,\ u^v = \varepsilon^{-2\left(\frac{\sin x}{x}\right)^2},$$

and the limit of this when $x = 0$ is ε^{-2}.

In the case 1^∞, if when $x = a$, $f(x) = 1$, $\phi(x) = \infty$, and $y = \{f(x)\}^{\phi(x)}$, $\log y = \dfrac{\log \phi(x)}{\dfrac{1}{f(x)}}$, and its limit is that of

$$\frac{\phi'(x)}{\phi(x)} \cdot \frac{-\overline{f(x)}\,\big]^2}{f'(x)} = -\phi'(a) \text{ limit of } \frac{\overline{f(x)}\,\big]^2}{f'(x)}.$$

Now $(x-a)f(x)$ takes the form $0 \times \infty$ when $x = a$, and its limit = limit of $\dfrac{x-a}{\dfrac{1}{f(x)}}$ = limit of $-\dfrac{\overline{f(x)}\,\big]^2}{f'(x)}$; therefore, if m be this limit, that of $\log y$ is $m\phi'(a)$, or that of y is $\varepsilon^{m\phi'(a)}$. Such forms can also generally be made to depend on the limit of $(1+z)^{\frac{m}{z}}$, when $z = 0$, which has been investigated in the first chapter, and whose limit is ε^m. For example,

$$(\cos x)^{\frac{m}{x^2}} = (\cos^2 x)^{\frac{m}{2x^2}} = (1 - \sin^2 x)^{\frac{m}{2x^2}} = \{(1 - \sin^2 x)^{\frac{m}{\sin^2 x}}\}^{\frac{\sin^2 x}{2x^2}},$$

and the limit when $x = 0$ is therefore $\varepsilon^{-\frac{m}{2}}$. Similarly the limits of $\left(\dfrac{\sin x}{x}\right)^{\frac{m}{x^2}}$, and $\left(\dfrac{\tan x}{x}\right)^{\frac{m}{x^2}}$ may be found, for

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \ldots = 1 - \frac{x^2}{6}(1+u),$$

where u vanishes with x, and

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3}(1+u'),$$

where u' vanishes with x; therefore

$$\left(\frac{\sin x}{x}\right)^{\frac{m}{x^2}} = \left\{1 - \frac{x^2}{6}(1+u)\right\}^{\frac{m}{x^2(1+u)}(1+u)},$$

and the limit is accordingly $\varepsilon^{-\frac{m}{6}}$, that of $\left(\frac{\tan x}{x}\right)^{\frac{m}{x^2}}$ being $\varepsilon^{\frac{m}{3}}$.

The two following limits, each of which may be made to depend on the other, are important: $x^m(\log x)^n$, $\frac{\varepsilon^{mx}}{x^n}$; the first when $x = 0$, the second when $x = \infty$, m, n being both positive. Now

$$x^m(\log x)^n = (x^{\frac{m}{n}}\log x)^n;$$

and limit

$$(x^{\frac{m}{n}}\log x)_{x=0} = \left(\frac{\log x}{x^{-\frac{m}{n}}}\right)_{x=0} = \left(\frac{\frac{1}{x}}{-\frac{m}{x}x^{-1-\frac{m}{n}}}\right) = \left(-\frac{n}{m}x^{\frac{m}{n}}\right)_{x=0} = 0.$$

So also $\left(\frac{\varepsilon^{mx}}{x^n}\right)_{x=\infty} = \left(\frac{m\varepsilon^{mx}}{nx^{n-1}}\right)_{x=\infty} = \left\{\frac{m^2\varepsilon^{mx}}{n(n-1)x^{n-2}}\right\}_{x=0} = \ldots$

until the index in the denominator becomes either 0 or negative, when the limit appears and is ∞.

These proofs are, however, unsound, as since the result in neither case is finite, the proof of the rule for finding $\frac{\infty}{\infty}$ fails. They are best proved by ordinary algebra, thus

$$\frac{\varepsilon^{mx}}{x^n} = \left(\frac{\varepsilon^{\frac{m}{n}x}}{x}\right)^n \equiv \left(\frac{\varepsilon^{px}}{x}\right)^n \text{ suppose, } p \text{ denoting } \frac{m}{n}.$$

Now $$\frac{\varepsilon^{px}}{x} = \frac{1}{x} + p + \frac{p^2}{\underline{|2}}x + \ldots;$$

and, therefore, when x is indefinitely increased, the sum of this series is indefinitely increased, or $\left(\frac{\varepsilon^{px}}{x}\right)_{x=\infty}$ is ∞;

and, therefore, also $\left(\frac{\varepsilon^{px}}{x}\right)^n$ is ∞, or $\frac{\varepsilon^{mx}}{x^n} = \infty$ when x. Now if we put $\varepsilon^x = \frac{1}{z}$, then when $x = \infty$, $z = 0$, and $\frac{\varepsilon^{mx}}{x^n}$ becomes $\left(\frac{1}{z}\right)^m \div (-\log z)^n$, or $\frac{1}{(-1)^n z^m (\log z)^n}$, and since the limit of this when $x = \infty$, and, therefore, when $z = 0$ is ∞, therefore that of $z^m (\log z)^n$ is 0, or the limit of $x^m (\log x)^n$ when $x = 0$ is 0.

In conclusion, it may be mentioned that any one who is familiar with the ordinary expansions in series of Algebra and Trigonometry, will find it in most cases easier to determine all such limits without the aid of the Differential Calculus at all. They should be reduced to the form $\frac{0}{0}$, which can always be done, as has been seen, and if this happen when $x = a$, put $a + z$ for x, and expand numerator and denominator in powers of z. The only failing case is of such functions as $\log x$, $\varepsilon^{-\frac{1}{x}}$, $\varepsilon^{-\frac{1}{x^2}}$... when $x = 0$, but these are quite as troublesome, if a strict proof is required, when the Calculus is used.

The following is an example given in Todhunter: $\frac{(\theta + \sin\theta - 4\sin\frac{1}{2}\theta)^4}{(3 + \cos\theta - 4\cos\frac{1}{2}\theta)^3}_{(\theta=0)}$; which would require 12 differentiations if solved by the rule; now

$$\theta + \sin\theta - 4\sin\tfrac{1}{2}\theta$$

$$= \theta + \theta - \frac{\theta^3}{6} + \ldots - 4\left(\frac{\theta}{2} - \frac{\theta^3}{48} + \ldots\right) = -\frac{\theta^3}{12} + \text{higher powers},$$

$$3 + \cos\theta - 4\cos\tfrac{1}{2}\theta$$

$$= 3 + 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \ldots - 4\left(1 - \frac{\theta^2}{8} + \frac{\theta^4}{16 \times 24} - \ldots\right)$$

$$= \frac{\theta^4}{32} + \text{higher powers},$$

hence the required limit is $\frac{(32)^3}{12^4} = \frac{2^{15}}{3^4 \times 2^8} = \frac{2^7}{3^4}$ or $\frac{128}{81}$.

I should guess no one had ever the patience to work this out according to rule.

If x, y are connected by an equation, so that each is what is called an *implicit* function of the other, and $F(x, y)=0$ be the equation, the equation for finding $\frac{dy}{dx}$ would, according to the general rule for differentiating complex functions, be $\left(\frac{dF}{dx}\right)+\left(\frac{dF}{dy}\right)\frac{dy}{dx}=0$, the brackets denoting *partial* differentiation. If this should reduce $\frac{dy}{dx}$ to the form $\frac{0}{0}$, for a pair of values $x=a$, $y=b$, which satisfy the equation $F(x, y)=0$, it is best to put $x=a+h$, $y=b+k$ in the equation, and find the value of the limit of $\frac{k}{h}$ directly from the equation. Since k is Δy, and h is Δx, this limit will be the value of $\frac{dy}{dx}$. The appearance of the form $\frac{0}{0}$ indicates a multiplicity of values; and if the curve $F(x, y)=0$ be drawn, the point will be what is called *singular*, there being two or more values of $\frac{dy}{dx}$ at the point, *i.e.* two or more tangents to the curve.

Thus, taking an example from Todhunter, if the equation be

$$(y^2-x^2)(x-1)(x-\tfrac{3}{2})-2(y^2+x^2-2x)^2=0,$$

$$\frac{dF}{dy}=2y(x-1)(x-\tfrac{3}{2})-8y(y^2+x^2-2x),$$

$$\frac{dF}{dx}=-2x(x-1)(x-\tfrac{3}{2})+(y^2-x^2)(2x-\tfrac{5}{2})-8(x-1)(y^2+x^2-2x),$$

both of which vanish when $x=1$, $y=1$. Put $x=1+x'$, $y=1+y'$, and we have

$$\{y'^2-x'^2+2(y'-x')\}x'(x'-\tfrac{1}{2})-2(y'^2+x'^2+2y')^2=0,$$

or $-x'(y'-x')-8y'^2+$ terms of higher dimensions which vanish compared with those retained when $x'=0$, $y'=0$.

Hence the equation for $\left(\frac{y'}{x'}\right)_0$ is $8z^2 + z - 1 = 0$, therefore $z = -\frac{1 \pm \sqrt{(33)}}{16}$.

This method should *always* be taken. The values of $\frac{dy}{dx}$ may be impossible; thus if the equation be $a^2y^2 = x^2(x^2 - a^2)$, the equation is satisfied by $x = 0$, $y = 0$, but we have $\frac{y^2}{x^2} = \frac{x^2}{a^2} - 1$, and therefore the limit of $\left(\frac{y}{x}\right)_{x=0}$ is $\pm\sqrt{(-1)}$. Such a point when the curve is traced should be conceived as an infinitely small loop or, in this case, circle, the limit of a finite one; for if the equation had been

$$a^2y^2 = x(x-b)(x^2-a^2),\ b < a,$$

there would be a loop of length b, closing up to a point when $b = 0$.

Maxima and Minima.

If $f(x)$ be any function of the independent variable x, and we conceive x to increase uniformly from $-\infty$ to $+\infty$, it will usually happen that $f(x)$ is not always increasing and not always decreasing, but that it sometimes does one and sometimes the other. If a, b, c be successive values of x, and if as x increases from $-\infty$ to a, $f(x)$ is always increasing, but from a to b, $f(x)$ is always decreasing, then $f(a)$ is said to be a maximum value of $f(x)$. If from $x = b$ to $x = c$, $f(x)$ is again always increasing, $f(b)$ is said to be a minimum value of $f(x)$ and so on. That is, a maximum value is greater and a minimum less than all *adjacent* values of $f(x)$, although a maximum value need not be the greatest of all, nor a minimum the least of all values of $f(x)$. (Of course, however, this may well be the case, and often is). Now if $f(x)$ be increasing as x increases, $f'(x)$ is positive; if $f(x)$ be decreasing as x increases, $f'(x)$ is negative. Hence the necessary and sufficient conditions for a maximum value of

$f(x)$ when $x=a$, are that $f'(x)$ shall change sign from positive to negative as x increases through the value a; and for a minimum that $f'(x)$ shall change sign from negative to positive as x increases through the value a. In general, the simplest method of finding such values is to observe this change of sign directly, but in some cases, and in especial, when $f(x)$ is what is called an *implicit* function of x, this change cannot easily be noted, and a different test to be explained afterwards must be applied. For all the functions which we have commonly to deal with, $f'(x)$ can only change sign by passing through the values 0 or ∞, otherwise it would be a *discontinuous* function, changing its value abruptly as x increases gradually. As simple examples of the test, consider the functions (1) x^3-3x+2, (2) $(x-a)^{\frac{2}{3}}$.

(1) $f(x)=x^3-3x+2$, $f'(x)=3(x+1)(x-1)$, when x has any value between $-\infty$ and -1, $x+1$, $x-1$ are both negative and $f'(x)$ positive, or $f(x)$ increases with x; but when x passes the value -1, and before it becomes so great as 1, $x+1$ is positive and $x-1$ negative; therefore $f'(x)$ is negative and $f(x)$ is decreasing. When x has passed the value 1, $f'(x)$ is again positive, and $f(x)$ again increases with x. Hence $f(-1)$ or 4 is a maximum and $f(1)$ or 0 a minimum value of $f(x)$.

(2) $f(x)=(x-a)^{\frac{2}{3}}$, $f'(x)=\frac{2}{3}.(x-a)^{-\frac{1}{3}}$, when $x<a$, $f'(x)$ is negative; and when $x>a$, $f'(x)$ is positive; hence, $f(x)$ decreases as x increases from $-\infty$ to a, and then increases, and $f(a)$ or 0 is a minimum value of $f(x)$.

To illustrate these results geometrically, draw the curves represented by the equations $y=x^3-3x+2$, $y=(x-a)^{\frac{2}{3}}$ respectively, *i.e.* to every distance x measured from 0 along the fixed straight line Ox, draw at right angles from its extremity a length $y=x^3-3x+2$ in (1), or to $(x-a)^{\frac{2}{3}}$ in (2). In (1) $OA=1$, $OB=-1$, $OD=2$, $BQ=4$, $BC=-1$, and the curve is somewhat as in fig. 12, so that y has the maximum value $BQ=4$, and its minimum value 0 at A, although on the branch beyond A there are an infinite number

of points for which the y is > 4, and similarly on the branch beyond C an infinite number where $y < 0$, *i.e.* is negative. So in general if the curve $y = f(x)$ be drawn, if y be a maximum or minimum where $f'(x)$ vanishes, the tangent at such points is parallel to the axis of x.

In (2) since $y^3 = (x-a)^2$, we see that if y be negative, x is impossible; if y be positive $x - a$ has two equal and opposite values, while at A, $(OA = a)$ $\frac{dy}{dx}$ is ∞, *i.e.* the tangent to the curve at A is perpendicular to OA. Hence the curve is as (fig. 13), the point A being what is called a *cusp*, and such a point always exists in the curve $y = f(x)$, if $f(x)$ has a maximum or minimum value when $f'(x) = \infty$.

As another example, take $f(x)$ or $y = (x+1)^4 (x-1)^6$; therefore

$$f'(x) = (x+1)^3 (x-1)^5 \{4(x-1) + 6(x+1)\}$$
$$= 2(x+1)^3 (5x+1)(x-1)^5.$$

Here $f'(x)$ vanishes when $x = -1$, $-\frac{1}{5}$, and 1; also $f'(x)$ is negative from $-\infty$ to -1, positive from -1 to $-\frac{1}{5}$, negative from $-\frac{1}{5}$ to 1, and afterwards always positive. Hence $f(-1)$ or 0 is a minimum value of $f(x)$, $f(-\frac{1}{5})$, or $\frac{8^4 \times 12^6}{10^{10}}$, or $1{\cdot}223\ldots$ is a maximum value, and $f(1)$ or 0 is again a minimum value. In this case we see at once that y is always positive for real values of x, and, therefore, that 0 must be a minimum value. The form of the curve (fig. 14) $y = f(x)$ in this case is somewhat as in the figure, where $OA = 1$, $OB = -1$, $OC = -\frac{1}{5}$, $CD = 1{\cdot}223\ldots$.

In general those values of x which make $f'(x) = 0$ or ∞ should be selected, and it should be observed whether the factor of $f'(x)$ corresponding to each value has its index odd or even (if that index be integral); if the index be odd $f'(x)$ must change sign as x passes through the corresponding value, and there will be *either* a maximum or minimum. But if the index be even (or of the form

$\frac{2p}{2q+1}$, *i.e.* $\frac{\text{even integer}}{\text{odd integer}}$) that factor itself can never change sign, and, therefore, $f'(x)$ will not change sign as x passes through that particular value, or there will be *neither* maximum nor minimum corresponding to that factor. Such values being struck out of the list, arrange the remainder in order of increasing magnitude $a, b, c \ldots$; observe the sign of $f'(x)$ when $x < a$, and, therefore, $x - a$ negative. If this sign be positive, $f'(x)$ will change from positive to negative as x increases through a, and $f(a)$ will be a maximum, $f(b)$ a minimum, $f(c)$ a maximum, and so on, until all the reserved factors have been taken account of. Thus, suppose

$$f'(x) = \frac{(x+3)^4 (x+2)^3 (x+1)^{\frac{2}{3}} x (x-1)^{\frac{3}{5}} (x-2)^2 (x-3)}{(x+4)^{\frac{1}{3}}},$$

so that $f'(x)$ vanishes when $x = -3, -2, -1, 0, 1, 2, 3$, and is ∞ when $x = -4$. Here the index of the factor $x + 3$ is 4, of $x + 1$ is $\frac{2}{3}$, and of $x - 2$ is 2, and none of these factors can change sign. The remaining critical values of x are $-4, -2, 0, 1, 3$, and when x is between $-\infty$ and -4, $f'(x)$ has 5 negative factors and is therefore negative. Hence $x = -4$ gives $f(x)$ a minimum, $x = -2$ gives $f(x)$ a maximum, $f(0)$ is a minimum, $f(1)$ a maximum, and $f(3)$ a minimum.

The above method gives the most satisfactory general rule for determining and distinguishing maximum and minimum values. In a very large proportion of geometrical applications, the nature of the question tells us at once whether the result is a maximum or a minimum, especially when there is only one solution of either sort. For instance, if PQ is a straight line drawn through a given point A and terminated by two given straight lines Ox, Oy, and the maximum or minimum length of PQ is required, it is manifest that when the straight line is drawn parallel to either Ox or Oy the length of the line

is infinite, hence there must be a minimum length in some intermediate position.

Another criterion for distinguishing maximum and minimum values is found as follows: $f(x)$ being a function of the independent variable x, $f(a)$ is said to be a maximum value of $f(x)$, if it is greater than all adjacent values of $f(x)$; *i.e.* if we put $a+h$ or $a-h$ for a, then the results are both less than $f(a)$ when h is taken sufficiently small.

Hence, for a maximum

$$f(a+h)-f(a), \text{ and } f(a-h)-f(a),$$

are *both* negative; and similarly for a *minimum* are both positive. But if $f(x)$ and its differential coefficients be *finite* for values of x between a and $a+h$,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a+\theta h);$$

or
$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{\lfloor 2}f''(a+\theta h);$$

$$f(a-h)-f(a)=-hf'(a)+\frac{h^2}{\lfloor 2}f''(a-\theta h).$$

Now, if $f'(a)$ be finite, since we can by diminishing h make the second term of each of these numerically less than the first, these two expressions must have *opposite* signs when h is taken sufficiently small. Hence, there can be neither maximum nor minimum unless $f'(a)=0$. If this be so, $f(a+h)-f(a)$, and $f(a-h)-f(a)$ will, when h is sufficiently small, have the same sign as $f''(a)$. But for a maximum both must be negative; therefore for a maximum value $f(a)$, we must have $f'(a)=0$, and $f''(a)$ negative. So, for a minimum, $f'(a)=0$, $f''(a)=$ a positive quantity. If $f''(a)=0$, and in general if all the differential coefficients up to the $(n-1)^{\text{th}}$ vanish, while $f^n(a)$ is finite, we have

$$f(a+h)-f(a)=\frac{h^n}{\lfloor n}f^n(a+\theta h),$$

$$f(a-h)-f(a)=\frac{(-h)^n}{\lfloor n}f^n(a-\theta h),$$

and when h is sufficiently small, these will be of opposite signs if n be odd, and there will be neither maximum nor minimum; but of the same sign if n be even, and that sign the same as the sign of $f^n(a)$, which must therefore be negative for a maximum and positive for a minimum.

Hence, for a maximum or minimum value $f(a)$ of $f(x)$, an *odd* number of the derived functions $f'(a)$, $f''(a)$, $f'''(a)$, ... must vanish, and if so, the sign of the first, which remains finite, will determine whether $f(a)$ is a maximum or minimum, viz. if it be negative, $f(a)$ is a maximum, if positive, $f(a)$ is a minimum.

As a simple example of the use of this method, suppose y is a function of x given by the equation $x^3 + y^3 - 3axy = 0$, hence $(y^2 - ax)\dfrac{dy}{dx} + x^2 - ay = 0$, and $\dfrac{dy}{dx}$ or $f'(x) = 0$ when $x^2 = ay$, and therefore

$$y^3 = 3axy - x^3 = 3axy - axy = 2axy,$$

therefore $y = 0$ or $y^2 = 2ax$. If we take $y = 0$, we have $x = 0$, and therefore $\dfrac{dy}{dx}$ becomes $\dfrac{0}{0}$ (one of the true values is 0 and gives a minimum), but taking $y^2 = 2ax$, therefore $y^4 = 4a^2x^2 = 4a^3y$, or $y^3 = 4a^3$, and therefore $x^3 = 2a^3$. To find the corresponding value of $f''(x)$ or $\dfrac{d^2y}{dx^2}$, differentiate the equation again, rejecting the terms involving $\dfrac{dy}{dx}$ as a factor since it is 0, and we have $(y^2 - ax)\dfrac{d^2y}{dx^2} + 2x = 0$, or since $y^2 = 2ax$, $ax\dfrac{d^2y}{dx^2} + 2x = 0$, or $\dfrac{d^2y}{dx^2} = -\dfrac{2}{a}$; therefore the value $4^{\frac{1}{3}}a$ is a maximum value of y. (To find whether the value $y = 0$ is a maximum or minimum, it is better to consider the equation directly; when y is small the term y^3 may be neglected compared with the others, and we have approximately $x^3 - 3axy = 0$, or $x^2 - 3ay = 0$ as an approximation form of the relation between x and y. Here we

should have $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=\frac{2}{3a}$, and therefore $y=0$ is a minimum value of y. For cases in which $\frac{dy}{dx}$ takes the form $\frac{0}{0}$, this method should always be used, putting $a+x$ for x, and $b+y$ for y, if a, b, be the values of x, y, for which $\frac{dy}{dx}=\frac{0}{0}\Big)$. The corresponding curve is represented in fig. (15).

In finding the maximum or minimum values of any quantity, care should be taken so to choose the variable in terms of which the quantity is expressed (that is, of which it is a *function*), that it may be capable of all values. For instance, if $u=F(z)$, and z be itself limited in value in any way, being a function of x, an independent variable capable of all values, then $u=F(z)$, and $\frac{du}{dz}=F'(z)$, but $\frac{du}{dx}=F'(z)\frac{dz}{dx}$, and although $\frac{du}{dz}=0$ when $F'(z)=0$, this may not furnish the true maximum or minimum values, which *may* occur when $\frac{dz}{dx}=0$, for which $\frac{du}{dx}=0$; for instance, suppose P (fig. 16) is any point on a fixed circle whose centre is C, O any other fixed point, and we seek the maximum and minimum values of OP. Take $CN=x$, $CP=a$, $CO=c$, then

$$OP^2=ON^2+NP^2=(c-x)^2+a^2-x^2$$
$$=c^2+a^2-2cx\equiv u,$$

then $\frac{du}{dx}=-2c$ which can never change sign. But in this case x is not capable of all values being limited to values between $-a$ and $+a$. If we make our independent variable the angle $OCP(=\theta)$, which may have any magnitude whatever, we have $u=a^2+c^2-2ca\cos\theta$, and $\frac{du}{d\theta}=2ca\sin\theta$, which changes from $-$ to $+$ as θ increases through 0, and from $+$ to $-$ as θ increases through π, hence $\theta=0$ gives

u a minimum, and $\theta=\pi$ gives u a maximum, as is obvious geometrically. In case of any failure by such a choice of independent variable, it is sufficient to look for the maximum and minimum values which the variable selected can have, for if $u=F(z)$ and $z=\phi(x)$, $\dfrac{du}{dx}=F'(z)\,\phi'(x)$, and $\dfrac{du}{dx}$ will *generally* change sign when $\phi'(x)$ does, that is when $\phi(x)$ or z is a maximum or minimum, the only exception being when $F'(z)$ changes sign at the same time as $\phi'(x)$.

Differential Calculus (4).

1. If, when $x=a$, $f(x)$ and $\phi(x)$ each $=0$, prove that the *limit* of the fraction $f(x)\div\phi(x)$, when x approaches the value a, is equal to the limit of $f'(x)\div\phi'(x)$. Also prove that the same rule holds if $f(a)=\infty$ and $\phi(a)=\infty$. Prove that the limits of $\dfrac{1}{x}\log\left(\dfrac{\varepsilon^x-1}{x}\right)$, and of its first derived function when $x=0$ are $\frac{1}{2}$, $\frac{1}{24}$ respectively.

2. Shew how to reduce fractions which assume the form $\infty\times 0$ or $\infty-\infty$, for a certain value of the independent variable, under the rule in (1). Prove that the limit, when $x=0$, of $x^m(\log x)^n$ is 0, and that of $\dfrac{1}{x^2}-\dfrac{1}{\tan^2 x}$ is $\frac{2}{3}$.

Find the limit of $\dfrac{1}{(\frac{1}{2}\pi-x)\cos x}-\dfrac{\sin x}{\cos^2 x}$ when $x=\frac{1}{2}\pi$.

3. Prove that the evaluation of the limits of functions which take the unmeaning forms of 1^∞, ∞°, 0° can be made to depend on functions which take the form $\dfrac{0}{0}$, and prove that the limit of u^v, where u and v each tend to 0, is always 1, provided the limit of $u\dfrac{dv}{dx}\div v\dfrac{du}{dx}$ is finite.

Find the limits of $(\sin\theta)^{\tan^2\theta}$, $(\tan\theta)^{\cos^2\theta}$, when $\theta=\frac{1}{2}\pi$, and of $(1-\cos\theta)^{\sin^2\theta}$ when $\theta=0$.

4. If $f(a)=1$ and $\phi(a)=\infty$, and the limit of $(x-a)\,\phi(x)$, when $x=a$, be m, then will the limit of $\{f(x)\}^{\phi(x)}$ be $\varepsilon^{mf'(a)}$.

5. Define a *maximum* or *minimum* value of $f(x)$ a function of the independent variable x, and prove that $f(a)$ will be a maximum value of $f(x)$, if $f'(x)$ change sign from positive to negative as x increases through the value a; and a minimum if $f'(x)$ change from negative to positive.

Find the maximum and minimum values of

$$(10x-6) \div (x+1)(x+3),$$

proving that the maximum value is 1, and the minimum 25; and explain how it happens that the minimum is greater than the maximum.

6. Find all the maximum and minimum values of $\dfrac{\tan^3 x}{\tan 3x}$, x lying between 0 and π, and explain how it is possible to have, as appears in this case, two successive minimum values $\left(x=\dfrac{3\pi}{8},\ x=\dfrac{5\pi}{8}\right)$.

$\{\dfrac{dy}{dx}=\dfrac{6\tan^3 x\cos 4x}{(\sin 3x)^2}$, and changes sign not only when $\sin x$ and $\cos 4x=0$, but also when $\cos x=0\}$.

7. Prove that the expression $x^{\frac{1}{x}}$ has a maximum value when $x=\varepsilon$.

8. A parabola is drawn having double contact with a given circle; prove that when the area included between this parabola and the tangent to the circle perpendicular to the axis of the parabola ($DPAP'D'$ in fig. 17) is a minimum, the latus rectum of the parabola is equal to the radius of the circle.

(If $4m$ be the latus rectum, a the radius of the circle,

$$a^2=CP^2=CM^2+MP^2=4m^2+4m\,.\,AM;$$

therefore

$$AB = AM + MC + CB = \frac{a^2 - 4m^2}{4m} + 2m + a = \frac{(a+2m)^2}{4m};$$

therefore

$$BD^2 = 4m.AB = (a+2m)^2, \text{ or } BD = a + 4m,$$

hence, area of parabola cut off by DD'

$$= \tfrac{4}{3} AB.BD = \frac{(a+2m)^3}{3m}$$

to be made a minimum by variation of m).

9. Find the minimum chord which can be drawn in a given ellipse normal to the ellipse.

[If $2a$, $2b$ be the axes of the ellipse, $a \cos\theta$, $b \sin\theta$ the point at which the chord is normal, and $2u$ its length, then

$$u^2 = \frac{a^2 b^2 (a^2 \sin^2\theta + b^2 \cos^2\theta)^3}{(a^4 \sin^2\theta + b^4 \cos^2\theta)^2},$$

and

$$\frac{1}{u}\frac{du}{d\theta} = (a^2 - b^2)\sin\theta\cos\theta\left\{\frac{3}{a^2\sin^2\theta + b^2\cos^2\theta} - \frac{2(a^2+b^2)}{a^4\sin^2\theta + b^4\cos^2\theta}\right\}$$

$$= \frac{(a^2-b^2)\sin\theta\cos\theta\,\{a^2(a^2-2b^2)\sin^2\theta - b^2(2a^2-b^2)\cos^2\theta\}}{(a^2\sin^2\theta + b^2\cos^2\theta)(a^4\sin^2\theta + b^4\cos^2\theta)};$$

and, therefore, $\dfrac{du}{d\theta}$ changes sign when $\theta = 0$, $\theta = \alpha$, $\theta = \tfrac{1}{2}\pi$, $\theta = \pi - \alpha$, where $\tan\alpha = \dfrac{b}{a}\sqrt{\left(\dfrac{2a^2-b^2}{a^2-2b^2}\right)}$, and since when θ is a small negative quantity $\dfrac{du}{d\theta}$ is positive, $\theta = 0$ gives a maximum, $\theta = \alpha$ a minimum, $\theta = \tfrac{1}{2}\pi$ a maximum, $\theta = \pi - \alpha$ a minimum, $\theta = \pi$ a maximum (same as $\theta = 0$), and so on. This assumes that $a^2 > 2b^2$, otherwise $\tan\alpha$ is impossible; and if $a^2 < 2b^2$, $\dfrac{du}{d\theta}$ changes sign only when $\theta = 0$ and $\theta = \tfrac{1}{2}\pi$, or the major axis is the only maximum and the minor axis the only minimum normal chord. See fig. (18), where PQ

is the minimum, corresponding to $\theta=\alpha$. AA' is a maximum, and BB' also a maximum, being greater than all adjacent ones. When $a^2=2b^2$, P, B, P' coincide, and BB' becomes the minimum].

Change of the Variable in a Differential Equation.

Differential equations may often be much simplified, and even reduced to a form in which we know the solution, by a change either of the independent or dependent variable. It is hardly worth while giving the formulæ for such changes, as they should always be effected, not by substituting the particular case in the formulæ, but by following the method indicated.

If an equation involve x, y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, ... and we wish to change the independent variable to z, z being a given function of x, (or x of z), we have first

$$\frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx}=\frac{dy}{dz}\div\frac{dx}{dz},$$

either of these may be used according as z is given in terms of x, or x of z.

Whence differentiating both sides, *with respect to* x, we have

$$\frac{d^2y}{dx^2}=\frac{dy}{dz}\,\frac{d^2z}{dx^2}+\frac{dz}{dx}\,\frac{d^2y}{dz^2}\,\frac{dz}{dx}=\frac{dy}{dz}\,\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2};$$

or
$$=\frac{\dfrac{dx}{dz}\,\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}\,\dfrac{d^2x}{dz^2}}{\left(\dfrac{dx}{dz}\right)^2}\,\frac{dz}{dx}=\frac{\dfrac{dx}{dz}\,\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}\,\dfrac{d^2x}{dz^2}}{\left(\dfrac{dx}{dz}\right)^3}.$$

So
$$\frac{d^3y}{dx^3}=\frac{dy}{dz}\,\frac{d^3z}{dx^3}+3\,\frac{d^2y}{dz^2}\,\frac{dz}{dx}\,\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^3\frac{d^3y}{dz^3},$$

or
$$=\frac{\dfrac{dx}{dz}\,\dfrac{d^3y}{dz^3}-\dfrac{dy}{dz}\,\dfrac{d^3x}{dz^3}}{\left(\dfrac{dx}{dz}\right)^4}-3\,\frac{\dfrac{d^2x}{dz^2}\left(\dfrac{dx}{dz}\,\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}\,\dfrac{d^2x}{dz^2}\right)}{\left(\dfrac{dx}{dz}\right)^5},$$

from which the *method* is sufficiently apparent.

Ex. $(a^2-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0$, where $x=a\cos\theta$,

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{-1}{a\sin\theta}\frac{dy}{d\theta};$$

therefore $$\frac{d^2y}{dx^2}=-\frac{d}{d\theta}\left(\frac{1}{a\sin\theta}\frac{dy}{d\theta}\right)\frac{d\theta}{dx},$$

or $$\frac{d^2y}{dx^2}=+\frac{1}{a^2\sin^2\theta}\frac{d^2y}{d\theta^2}-\frac{\cos\theta}{a^2\sin^3\theta}\frac{dy}{d\theta};$$

therefore $$(a^2-x^2)\frac{d^2y}{dx^2}=a^2\sin^2\theta\frac{d^2y}{dx^2}=\frac{d^2y}{d\theta^2}-\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}$$

$$-x\frac{dy}{dx}=-a\cos\theta\frac{-1}{a\sin\theta}\frac{dy}{d\theta}=\cot\theta\frac{dy}{d\theta};$$

therefore $$(a^2-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=\frac{d^2y}{d\theta^2},$$

and the transformed equation is $\dfrac{d^2y}{d\theta^2}+m^2y=0$.

We will now use the other method, putting $\theta=\cos^{-1}\left(\dfrac{x}{a}\right)$;

therefore $$\frac{dy}{dx}=\frac{dy}{d\theta}\frac{d\theta}{dx}=\frac{-1}{\sqrt{(a^2-x^2)}}\frac{dy}{d\theta};$$

therefore $$\sqrt{(a^2-x^2)}\frac{dy}{dx}=-\frac{dy}{d\theta};$$

therefore

$$\sqrt{(a^2-x^2)}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{(a^2-x^2)}}\frac{dy}{dx}=-\frac{d^2y}{d\theta^2}\frac{d\theta}{dx}=\frac{1}{\sqrt{(a^2-x^2)}}\frac{d^2y}{d\theta^2},$$

or $$(a^2-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=\frac{d^2y}{d\theta^2};$$

so that in this case the second method gives the result more readily, but there is not usually much to choose between them.

As an example of changing both dependent and independent variables, take the equation

$$(1+x^2)\frac{d^2y}{dx^2}+2(n+1)x\frac{dy}{dx}+n(n+1)y=0,$$

and let (1) $x = \tan\theta$, (2) $y = z\cos^n\theta$, then

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\cdot\frac{1}{1+x^2}, \text{ or } (1+x^2)\frac{dy}{dx} = \frac{dy}{d\theta};$$

therefore $$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{d^2y}{d\theta^2}\frac{1}{1+x^2},$$

or $$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = \frac{d^2y}{d\theta^2};$$

also $$2nx(1+x^2)\frac{dy}{dx} = 2n\frac{dy}{d\theta}\tan\theta,$$

therefore

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2(n+1)x(1+x^2)\frac{dy}{dx} = \frac{d^2y}{d\theta^2} + 2n\tan\theta\frac{dy}{d\theta},$$

or $$(1+x^2)\frac{d^2y}{dx^2} + 2(n+1)x\frac{dy}{dx} = \frac{d^2y}{d\theta^2}\cos^2\theta + 2n\sin\theta\cos\theta\frac{dy}{d\theta},$$

and the equation after the first change is

$$\frac{d^2y}{d\theta^2}\cos^2\theta + 2n\sin\theta\cos\theta\frac{dy}{d\theta} + n(n+1)y = 0.$$

Next, $\dfrac{dy}{d\theta} = \cos^n\theta\dfrac{dz}{d\theta} - nz\cos^{n-1}\theta\sin\theta$, for $y = z\cos^n\theta$,

$$\frac{d^2y}{d\theta^2} = \cos^n\theta\frac{d^2z}{d\theta^2} - 2n\cos^{n-1}\theta\sin\theta\frac{dz}{d\theta}$$
$$+ nz\{(n-1)\cos^{n-2}\theta\sin^2\theta - \cos^n\theta\};$$

therefore $\cos^2\theta\dfrac{d^2y}{d\theta^2} + 2n\sin\theta\cos\theta\dfrac{dy}{d\theta}$

$$= \cos^{n+2}\theta\frac{d^2z}{d\theta^2} - 2n\cos^{n+1}\theta\sin\theta\frac{dz}{d\theta} + nz\cos^n\theta(n\sin^2\theta - 1)$$
$$+ 2n\cos^{n+1}\theta\sin\theta\frac{dz}{d\theta} - 2n^2z\cos^n\theta\sin^2\theta;$$

also $$n(n+1)y = n(n+1)z\cos^n\theta;$$

therefore, adding,

$$0 = \cos^{n+2}\theta\frac{d^2z}{d\theta^2} + n^2z\cos^n\theta(1-\sin^2\theta) = \cos^{n+2}\theta\left\{\frac{d^2z}{d\theta^2} + n^2z\right\},$$

and the final equation is

$$\frac{d^2z}{d\theta^2} + n^2z = 0,$$

whence $z = A\cos n\theta + B\sin n\theta$, so that the complete solution of the original equation is

$$y = (1+x^2)^{-\frac{1}{2}n}\{A\cos(n\tan^{-1}x) + B\sin(n\tan^{-1}x)\}.$$

Differential equations involving $x\dfrac{dy}{dx}$, $x^2\dfrac{d^2y}{dx^2}\dots$, are much simplified by taking $x = \varepsilon^z$, for

$$\frac{dy}{dx} = \frac{dy}{dz} \div \frac{dx}{dz} = \frac{1}{x}\frac{dy}{dz}, \text{ i.e. } x\frac{dy}{dx} = \frac{dy}{dz};$$

therefore $$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{1}{x};$$

therefore $$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right)y;$$

therefore $$x^2\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\frac{dy}{dz},$$

or $$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} = \left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz}\right)^2 y,$$

$$x^3\frac{d^3y}{dx^3} = \left(\frac{d}{dz} - 1\right)\frac{d^2}{dz^2}y - 2\left(\frac{d}{dz} - 1\right)\frac{d}{dz}y$$

$$= \frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)y,$$

and so on, the general formula being

$$x^n\frac{d^ny}{dx^n} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\dots\left(\frac{d}{dz} - n + 1\right)y.$$

Geometrical Applications.

If x, y be any point P on a curve, $x + \Delta x$, $y + \Delta y$ a contiguous point Q, (X, Y) coordinates of any point R on the straight line PQ, then, as in fig. 19,

$$\frac{lR}{Pl} = \frac{nQ}{Pn}, \text{ or } \frac{Y - y}{X - x} = \frac{\Delta y}{\Delta x},$$

i.e. the *equation* of the chord PQ is

$$Y - y = \frac{\Delta y}{\Delta x}(X - x),$$

and the equation of the tangent at P is the limit of this when Q moves up to P, or is

$$Y - y = \frac{dy}{dx}(X - x),$$

and the equation of the normal at P is therefore

$$(Y - y)\frac{dy}{dx} + X - x = 0.$$

Let ψ be the angle which the tangent at P makes with the axis of x, then $\tan\psi = \frac{dy}{dx}$; also $\cos\psi =$ limit of $\frac{Pn}{PQ} =$ limit of $\frac{\Delta x}{PQ}$, but the limit of $\frac{\text{chord } PQ}{\text{arc } PQ}$ is 1, or if the arc of the curve measured from any fixed point up to P be s, and arc $PQ = \Delta s$,

$$\cos\psi = \text{limit } \frac{\Delta x}{\Delta s} \times \frac{\text{arc } PQ}{\text{chord } PQ} = \frac{dx}{ds},$$

and similarly $$\sin\psi = \frac{dy}{ds}.$$

If the curve be referred to polar coordinates, $SP = r$, $ASP = \theta$, Q (fig. 20) a neighbouring point whose coordinates are $SQ = r + \Delta r$, $ASQ = \theta + \Delta\theta$, then

$$\cos SQP = \frac{QL}{QP} = \frac{r + \Delta r - r\cos\Delta\theta}{\Delta s} \cdot \frac{\text{arc } QP}{\text{chord } QP}.$$

Now $$(1 - \cos\Delta\theta) = 2\sin^2\frac{\Delta\theta}{2},$$

and $$\frac{1 - \cos\Delta\theta}{\Delta s} = \frac{2\sin\frac{\Delta\theta}{2}}{\Delta s} \cdot \sin\frac{\Delta\theta}{2},$$

or vanishes in the limit; therefore, if ϕ be the angle SPT which the tangent at P makes with the radius vector,

$$\cos\phi = \frac{dr}{ds},$$

and similarly $\sin\phi = \text{limit of } \sin SQP = \text{limit of } \frac{LP}{QP}$

$$= \text{limit of } r\frac{\sin\Delta\theta}{\Delta s}\cdot\frac{\text{arc } QP}{\text{chord } QP} = r\frac{d\theta}{ds},$$

and $$\tan\phi = r\frac{d\theta}{dr}.$$

(Of course since $x = r\cos\theta$, $y = r\sin\theta$, these results could be deduced from the former in x and y, and it will be a useful exercise so to obtain them).

Hence we have the equations

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + \left(r\frac{d\theta}{ds}\right)^2;$$

or if t be any other variable in terms of which we can express x, y, r, θ,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2.$$

It is often most convenient to express x and y the coordinates of any point of a curve in terms of ψ, the angle which the tangent makes with the axis of x (or any other fixed straight line). Thus, if the curve be the parabola

$$y^2 = 4ax, \quad \frac{dy}{dx} = \frac{2a}{y} = \tan\psi,$$

or $$y = 2a\cot\psi, \quad x = a\cot^2\psi.$$

In this case it is better to put $\frac{1}{2}\pi - \psi$ for ψ, so that ψ will be the angle which the tangent makes with the axis of y, and we shall now have x, y, ψ start together; and

$$x = a\tan^2\psi, \quad y = 2a\tan\psi;$$

therefore $$\frac{dx}{d\psi} = 2a\tan\psi\sec^2\psi = \frac{2a\sin\psi}{\cos^3\psi},$$

$$\frac{dy}{d\psi} = \frac{2a}{\cos^2\psi}; \text{ therefore } \frac{ds}{d\psi} = \frac{2a}{\cos^3\psi},$$

$\left(s \text{ and } \psi \text{ increasing together } \frac{ds}{d\psi} \text{ is positive}\right)$, from which the arc to any point may be found,

$$s = 2a\int\frac{d\psi}{\cos^3\psi} = 2a\int\frac{1}{\cos\psi}\,d\tan\psi$$

$$= 2a\left(\frac{\sin\psi}{\cos^2\psi} - \int\tan\psi\,\frac{\sin\psi}{\cos^2\psi}\,d\psi\right)$$

$$= 2a\left\{\frac{\sin\psi}{\cos^2\psi} - \int\frac{1-\cos^2\psi}{\cos^3\psi}\,d\psi\right\};$$

therefore

$$2s = 2a\left\{\left(\frac{\sin\psi}{\cos^2\psi} + \int\frac{d\psi}{\cos\psi}\right) = 2a\left(\frac{\sin\psi}{\cos^2\psi} + \log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)\right\}$$

$$+ (C=0)\ (s=0 \text{ when } \psi=0),$$

or s the arc measured from the vertex to any point is

$$a\left\{\frac{\sin\psi}{\cos^2\psi} + \log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right\}.$$

Thus the arc from the vertex to the end of the latus rectum

$$= a\,\{\sqrt{(2)} + \log(\sqrt{2}+1)\}.$$

Again take the catenary

$$y = \frac{c}{2}\left(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}}\right),\quad \frac{dy}{dx} = \frac{1}{2}\left(\varepsilon^{\frac{x}{c}} - \varepsilon^{-\frac{x}{c}}\right) = \tan\psi,$$

therefore $\varepsilon^{\frac{x}{c}} = \tan\psi + \sec\psi$; also $\varepsilon^{-\frac{x}{c}} = \sec\psi - \tan\psi$,

and $$y = \frac{c}{\cos\psi},\quad x = c\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right),$$

$$\frac{dy}{d\psi} = \frac{c\sin\psi}{\cos^2\psi} = \frac{dy}{ds}\,\frac{ds}{d\psi} = \sin\psi\,\frac{ds}{d\psi};$$

therefore $$\frac{ds}{d\psi} = \frac{c}{\cos^2\psi},$$

and $s = c\tan\psi$ measured from the lowest point of the curve.

Next take the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; any point on this curve may be represented by

$$x = a\cos^3\theta,\quad y = a\sin^3\theta;$$

therefore

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta=\tan\psi;$$

therefore $\theta=\pi-\psi$, or if the tangent at P (fig. 21) meet the axis of x in L, $\theta=\pi-\angle PLx=\angle PLO$. Hence

$$OL=ON+NL=a\cos^3\theta+a\sin^3\theta\cot\theta$$
$$=a\cos\theta\,(\cos^2\theta+\sin^2\theta)=a\cos\theta,$$

or $LM=a$, the most important property of the curve, that the part of the tangent intercepted between the axes is of constant length. Again $x=-a\cos^3\psi$; therefore

$$\frac{dx}{d\psi}=3a\sin\psi\cos^2\psi=\frac{dx}{ds}\frac{ds}{d\psi}=\cos\psi\frac{ds}{d\psi},$$

or $$\frac{ds}{d\psi}=3a\sin\psi\cos\psi;\ \text{therefore } s=3a\frac{\sin^2\psi}{2},$$

measuring from the point where $\psi=0$; *i.e.* $x=a$, $y=0$. This curve is generated by any point on a circle of radius $\frac{a}{4}$ which rolls within a circle of radius a, giving an equal branch in each quadrant, the length of which is $\frac{3a}{2}$; or the length of the whole curve is $6a$.

Curvature, Radius of Curvature.

In any circle the radius is equal to $\frac{s}{\psi}$ if s be any arc whatever, and ψ the angle through which the tangent turns as the arc s is traversed; hence the *curvature* of any circle, being inversely as the radius, is proportional to $\frac{\psi}{s}$, and we may take this conveniently as the measure of the curvature; and the arc s or the angle ψ may be as large or as small as we choose. So, in general, in any curve, s, ψ having the same meaning, the *average* curvature of the arc s is $\frac{\psi}{s}$, since this would give the same total deflection of the tangent in passing over the arc s, hence the average

curvature of a small arc Δs at the end of the former arc will be $\frac{\Delta\psi}{\Delta s}$, and if we diminish Δs indefinitely, we have, finally, the curvature at the end of an arc s of any curve is $\frac{d\psi}{ds}$, or the *radius* of curvature, *i.e.* the radius of the circle which has the same curvature as the curve is $\frac{ds}{d\psi}$. This can, of course, be expressed in terms of (x, y), since

$$\tan\psi = \frac{dy}{dx}, \text{ and } \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

$$\sec^2\psi\frac{d\psi}{dx} = \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{d\psi}{dx};$$

hence the curvature is

$$\frac{d\psi}{ds} = \frac{d\psi}{dx} \div \frac{ds}{dx} = \frac{d^2y}{dx^2} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$$

In forming this we have assumed that $\frac{ds}{dx}$, $\frac{ds}{d\psi}$ are positive, which will apply to a curve with its convexity towards the axis of x; if $\frac{ds}{dx}$ be negative, as in a curve whose concavity is towards the axis of x, the curvature is $-\frac{d^2y}{dx^2} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$. Some prefer giving algebraic sign to the curvature and consider it positive when it turns from the axis of x, and negative when it turns towards the axis of x. This is however of slight importance.

Another convenient expression for the radius of curvature is $p + \frac{d^2p}{d\psi^2}$, where p is the perpendicular from a fixed point on the tangent, and ψ as before. In fig. 22 OY is perpendicular on the tangent PT at P, $ON = x$, $NP = y$, NM, NZ perpendiculars on the tangent and on OY, then

$$p = OY = OZ - YZ = OZ - MN = x\sin\psi - y\cos\psi,$$

therefore

$$\frac{dp}{d\psi} = x\cos\psi + y\sin\psi$$

$$+ \left(\frac{dx}{d\psi}\sin\psi - \frac{dy}{d\psi}\cos\psi \text{ which} = 0 \text{ for } \frac{dy}{dx} = \tan\psi\right)$$

$$= ZN + MP = YP = \text{perpend. from } O \text{ on the normal at } P,$$

$$\frac{d^2p}{d\psi^2} = -x\sin\psi + y\cos\psi$$

$$+ \left\{\frac{dx}{d\psi}\cos\psi + \frac{dy}{d\psi}\sin\psi \text{ which } = \frac{ds}{d\psi}(\cos^2\psi + \sin^2\psi)\right.$$

$$\left. = \frac{ds}{d\psi} \text{ since } \frac{dx}{ds} = \cos\psi \text{ and } \frac{dy}{ds} = \sin\psi\right\}$$

$$= -p + \frac{ds}{d\psi};$$

therefore $$p + \frac{d^2p}{d\psi^2} = \frac{ds}{d\psi}.$$

Hence we have $s = \dfrac{dp}{d\psi} + \int p\,d\psi$, and since $\dfrac{dp}{d\psi}$ is the perpendicular on the normal at P, we shall have for the whole arc of any *closed* oval curve without singular points, so that the curve re-enters as soon as ψ is increased by 2π, that the whole perimeter $= \displaystyle\int_0^{2\pi} p\,d\psi$, since $\dfrac{dp}{d\psi}$ will have the same value when $\psi = 0$ as when $\psi = 2\pi$.

Since the normal at P (fig. 23), as P occupies successive positions on the curve, will touch some curve or other, let Q be its point of contact, QY' perpendicular to PQ will be the normal at Q to this curve, and since we have seen that $\dfrac{dp}{d\psi}$ is the perpendicular from O on PQ, if we call this p', the perpendicular from O on the normal at Q will be $\dfrac{dp'}{d\psi'}$, but $\psi' = \frac{1}{2}\pi + \psi$, and $p' = \dfrac{dp}{d\psi}$; therefore the per-

pendicular from O on the normal at $Q = \dfrac{d^2 p}{d\psi^2} = OY'$; therefore $YY' = p + \dfrac{d^2 p}{d\psi^2} = PQ$, or PQ = radius of curvature at P, or Q is the centre of curvature at P. This proves that the centre of curvature at P is the limiting position of the point of intersection of the normals at P, P' when P' moves up to P.

Differential Coefficients of Areas, Volumes, Surfaces.

(1) Let U be the area intercepted between a curve $y = \phi(x)$, the axis of x, the ordinate at any point P (fig. 24), and some fixed ordinate aA,

$$OM = x,\ MP = \phi(x),\ ON = x + \Delta x,\ \ NQ = \phi(x + \Delta x),$$

then $$U = AaMP,\ U + \Delta U = AaNQ;$$

therefore $$\Delta U = \text{area } PMNQ,$$

which for all forms of the curve, provided $\phi(x)$ is finite, will lie between $\phi(x)\Delta x$, $\phi(x+\Delta x)\Delta x$, when Δx is taken sufficiently small; or $\dfrac{\Delta U}{\Delta x}$ lies between $\phi(x)$, and $\Phi(x + \Delta x)$, or $\dfrac{dU}{dx} = \phi(x)$, or $U = \displaystyle\int_a^x \phi(x)\,dx$ where $Oa = a$.

If U be the area included between the curve, the radius vector to a point P (fig. 25) and some fixed radius vector OA, where

$$xOA = \alpha,\ OP = r,\ XOP = \theta,\ OQ = r + \Delta r,\ XOQ = \theta + \Delta\theta,$$

$$U = \text{area } AOP,\ U + \Delta U = \text{area } AOQ,$$

$$\Delta U = \text{area } POQ = \text{triangle } POQ \text{ in the limit,}$$

(this means not that they are equal because both vanish, but that their limiting ratio is one of equality),

$$= \tfrac{1}{2}(r + \Delta r)\, r \sin \Delta\theta;$$

therefore $$\frac{\Delta U}{\Delta \theta} = \tfrac{1}{2} \frac{\sin \Delta \theta}{\Delta \theta} . r (r + \Delta r),$$

$$\frac{dU}{d\theta} = \tfrac{1}{2} r^2, \text{ or } U = \tfrac{1}{2} \int_a^\theta r^2 d\theta,$$

r being a known function of θ.

(2) Let V be the volume generated by the revolution of AP (fig. 24) about the axis of x, then ΔV will be the volume generated by the revolution of area $PMNQ$ about the axis of x, which will always, when Δx is small enough, lie between the two cylinders whose common axis is MN, and radii respectively MP, NQ, or between

$$\pi (y^2 \Delta x) \text{ and } \pi (y + \Delta y)^2 \Delta x,$$

or $$\frac{\Delta V}{\Delta x} \text{ lies between } \pi y^2 \text{ and } \pi (y + \Delta y)^2,$$

$$\frac{dV}{dx} = \pi y^2, \quad V = \pi \int_a^x \{f(x)\}^2 dx.$$

(3) If S be the area of the surface generated by the revolution of AP (fig. 26) about the axis of x, then ΔS will be the area generated by the revolution of the arc PQ or Δs, and since each point of this arc is at a distance from the axis of x, which lies between y and $y + \Delta y$, therefore ΔS must lie between

$$\Delta s . 2\pi y, \text{ and } \Delta s . 2\pi (y + \Delta y),$$

or $$\frac{\Delta S}{\Delta x} \text{ lies between } 2\pi y \frac{\Delta s}{\Delta x} \text{ and } 2\pi (y + \Delta y) \frac{\Delta s}{\Delta x};$$

therefore $$\frac{dS}{dx} = 2\pi y \frac{ds}{dx};$$

or $$S = 2\pi \int f(x) . \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx$$

$$= 2\pi \int_a^x \sqrt{[1 + \{f'(x)\}^2]} . f(x) \, dx.$$

(4) Let U be the area generated by the revolution of the area AOP (fig. 26) about OA, let fall PM per-

pendicular on OA, and produce MP to meet SQ in R, then $SP=r$, $AOP=\theta$, $SQ=r+\Delta r$, $ASQ=\theta+\Delta\theta$, and the volume generated by the revolution of POQ is δU, but the volume generated by the area PQR vanishes in the limit compared with that generated by POR, hence $\Delta U=$ ultimately the volume generated by $POQ=$ volume generated by $MOR-$ volume generated by MOP

$$= -\tfrac{1}{3}\pi r^2 \sin^2\theta . r \cos\theta + \tfrac{1}{3}\pi \{r\cos\theta . \tan(\theta+\Delta\theta)\}^2 r\cos\theta$$

$$= \tfrac{1}{3}\pi r^3 \cos\theta\,(-\sin^2\theta + \cos^2\theta \tan^2\theta')$$

$$= \tfrac{1}{3}\pi r^3 \cos\theta . \frac{\sin(\theta'-\theta)\sin(\theta'+\theta)}{\cos^2\theta'}$$

$$= \tfrac{1}{3}\pi r^3 \cos\theta \frac{\sin\Delta\theta\,(\sin 2\theta+\Delta\theta)}{\cos^2\theta'};$$

therefore $$\frac{\Delta U}{\Delta\theta} = \tfrac{1}{3}\pi r^3 \cos\theta . \frac{\sin(2\theta+\Delta\theta)}{\cos^2(\theta+\Delta\theta)} . \frac{\sin\Delta\theta}{\Delta\theta},$$

$$\frac{dU}{d\theta} = \tfrac{1}{3}\pi r^3 \frac{\sin 2\theta}{\cos\theta} = \tfrac{2}{3}\pi r^3 \sin\theta;$$

therefore $$U = \tfrac{2}{3}\pi \int_0^\theta r^3 \sin\theta d\theta,$$

r being a known function of θ.

As an example of the use of all these formulæ, take the curve $9ay^2=(x+a)(x-2a)^2$, which is of the form given, (fig. 27)

$$18ay\frac{dy}{dx} = (x-2a)^2 + 2(x+a)(x-2a) = 3(x-2a)x,$$

$$\frac{dy}{dx} = \frac{x(x-2a)}{6ay} = \frac{x(x-2a)}{2\sqrt{\{a(x+a)(x-2a)\}^2}},$$

or $$\frac{dy}{dx} = \pm\frac{x}{2\sqrt{\{a(x+a)\}}} = \frac{x}{2\sqrt{\{a(x+a)\}}},$$

if we take the lower branch BAP, or

$$\tan\psi = \frac{x}{2\{a(x+a)\}};$$

therefore $$\sin^2\psi = \frac{x^2}{4a(x+a)+x^2} = \left(\frac{x}{x+2a}\right)^2,$$

or $\quad \dfrac{x}{x+2a} = \pm \sin\psi = \sin\psi$ on the same branch as before,

when $\quad (x=-a,\ \psi=-\tfrac{1}{2}\pi,\ x=0,\ \psi=0),$

$$x = \frac{2a\sin\psi}{1-\sin\psi} = \frac{2a}{1-\sin\psi} - 2a,$$

therefore $\dfrac{dx}{d\psi} = \dfrac{2a\cos\psi}{(1-\sin\psi)^2} = \dfrac{dx}{ds}\cdot\dfrac{ds}{d\psi} = \cos\psi\,\dfrac{ds}{d\psi};$

therefore $$\frac{ds}{d\psi} = \frac{2a}{(1-\sin\psi)^2};$$

therefore $\quad s = 2a\displaystyle\int_{-\frac{1}{2}\pi}^{\psi}\frac{d\psi}{(1-\sin\psi)^2} = 2a\int_0^{\theta}\frac{d\theta}{(1+\cos\theta)^2},$

if $\theta = \tfrac{1}{2}\pi + \psi$, or

$$\text{arc } VBAP = 2a\int\frac{d\tan\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = a\int(1+\tan^2\tfrac{1}{2}\theta)\,d\tan\tfrac{1}{2}\theta$$

$$= a\left(\tan\tfrac{1}{2}\theta + \tfrac{1}{3}\tan^3\tfrac{1}{2}\theta\right)$$

$$= a\tan\tfrac{1}{2}\theta.\left(1+\tfrac{1}{3}\,\frac{1-\cos\theta}{1+\cos\theta}\right) = \frac{2a}{3}\,\frac{\sin\theta}{1+\cos\theta}\cdot\frac{2+\cos\theta}{1+\cos\theta}$$

$$= \frac{2a}{3}\,\frac{\cos\psi\,(2-\sin\psi)}{(1-\sin\psi)^2}.$$

Now $$MP = x = \frac{2a\sin\psi}{1-\sin\psi};$$

therefore $$MK = MP\cot\psi = \frac{2a\cos\psi}{1-\sin\psi},$$

and $$OM = \sqrt{\left(\frac{x+a}{9a}\right)}.(x-2a)$$

$$= \frac{2a}{3}\sqrt{\left(\frac{1+\sin\psi}{1-\sin\psi}\right)}\cdot\frac{2\sin\psi-1}{1-\sin\psi}$$

$$= \frac{2a}{3}\,\frac{\cos\psi\,(2\sin\psi-1)}{(1-\sin\psi)^2};$$

therefore $\quad OK = \dfrac{2a\cos\psi}{3(1-\sin\psi)^2}\{3(1-\sin\psi)+(2\sin\psi-1)\}$

$$= \frac{2a\cos\psi\,(2-\sin\psi)}{3(1-\sin\psi)^2} = \text{arc } VBAP.$$

Also $$PK = \frac{MP}{\sin\psi} = \frac{2a}{1-\sin\psi} = MP + 2a,$$

i.e. the normal intercepted by the axis of y exceeds the abscissa by a constant quantity. The radius of curvature

$$= \frac{2a}{(1-\sin\psi)^2} = \frac{PK^2}{2a} \text{ or } \propto PK^2.$$

For the area of any part of the loop,

$$A = \tfrac{2}{3}\int_{-a}^{x} \sqrt{\left(\frac{x+a}{a}\right)}(2a-x)\,dx, \text{ or if } x+a = az^2,$$

$$A = \tfrac{2}{3}a^2\int_0^z z(3-z^2)\,2zdz = \frac{4a^2}{3}\left(z^3 - \frac{z^5}{5}\right)$$

$$= \frac{4a^2}{3}\sqrt{\left(\frac{x+a}{a}\right)}\cdot\left(\frac{x+a}{a} - \overline{\frac{x+a}{5a^2}}\right]^2 = \tfrac{4}{15}\frac{(x+a)^{\frac{3}{2}}}{\sqrt{(a)}}(4a-x),$$

hence the area of the whole loop is $\frac{8\sqrt{3}}{5}a^2$.

The volume generated by the revolution of the loop

$$= \frac{\pi}{9a}\int_{-a}^{2a}(x+a)(x-2a)^2\,dx = \frac{\pi}{9}a^3\int_0^3 z(3-z)^2\,dz$$

$$= \frac{\pi a^3}{9}\left(\frac{81}{2} - \frac{6.27}{3} + \frac{81}{4}\right) = \pi a^3\left(\frac{9}{2} - 6 + \frac{9}{4}\right) = \frac{3}{4}\pi a^3.$$

Also the area of the surface generated by the revolution of the loop

$$= 2\pi\int_{-a}^{2a} y\frac{ds}{dx}dx,$$

and $\dfrac{ds}{dx} = \dfrac{2a+x}{2\sqrt{\{a(x+a)\}}}$, $y = \dfrac{1}{3}\sqrt{\left(\dfrac{x+a}{a}\right)}\cdot(2a-x)$;

therefore $$y\frac{ds}{dx} = \frac{1}{6}\cdot\frac{(4a^2-x^2)}{a};$$

therefore

$$\text{area} = 2\pi\frac{a^2}{6}\int_{-1}^{2}(4-z^2)\,dz = \pi\frac{a^2}{6}\{12-3\} = 3\pi a^2.$$

Coordinates of the centre of curvature are

$$x - \frac{ds}{d\psi}\sin\psi = \frac{2a\sin\psi}{1-\sin\psi} - \frac{2a\sin\psi}{(1-\sin\psi)^2} = \frac{-2a.\sin^2\psi}{(1-\sin\psi)^2},$$

$$y + \frac{ds}{d\psi}\cos\psi = \frac{2a}{3}\,\frac{\cos\psi\,(2\sin\psi - 1)}{(1-\sin\psi)^2} + \frac{2a\cos\psi}{(1-\sin\psi)^2}$$

$$= \frac{4a}{3}\,\frac{\cos\psi}{(1-\sin\psi)^2}\,(1+\sin\psi),$$

or $$X = -\frac{2a\sin^2\psi}{(1-\sin\psi)^2},\quad Y^2 = \frac{16a^2}{9}\left(\frac{1+\sin\psi}{1-\sin\psi}\right)^3,$$

from which the equation of the locus of the centre of curvature is readily found,

$$\sqrt{\left(\frac{-2X}{a}\right)} = \frac{2\sin\psi}{1-\sin\psi};$$

therefore $$1 + \sqrt{\left(\frac{-2X}{a}\right)} = \left(\frac{3Y}{4a}\right)^{\frac{2}{3}};$$

$$\left(\frac{3Y}{4a}\right)^{\frac{2}{3}} + \left(\frac{-2X}{a}\right)^{\frac{1}{2}} = 1.$$

The curve represented by this equation is also drawn in fig. 27.

Differential Calculus. (5).

1. The tangent to any curve at a point (x, y) makes with the axis of x an angle whose tangent is $\frac{dy}{dx}$, cosine $\frac{dx}{ds}$, and sine $\frac{dy}{ds}$, s being the arc of the curve measured from a fixed point to the point (x, y); also the tangent makes with the radius vector at a point (r, θ) an angle whose tangent is $r\frac{d\theta}{dr}$, cosine $\frac{dr}{ds}$, and sine $r\frac{d\theta}{ds}$.

2. Prove that the length of the tangent at any point of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the axes of coordinates is always a.

3. The curve $9ay^2 = (x+a)(x-2a)^2$ is such, that if at any point P the normal meet the axis of y in K, and PM

be perpendicular to the axis of y,

$$PK = NP + 2a\text{; and if } KPN = \theta,\ NP = \frac{2a\cos\theta}{1-\cos\theta}.$$

4. If s be the arc of a curve measured from any fixed point to the point (x, y) or (r, θ), then will

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,\quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

Prove that the whole arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $6a$, and that of the curve in (3) measured from the vertex (VBP) is equal to OK.

5. Find the differential coefficients of the area of any curve (1) included between the curve, the axis of x, and two ordinates; (2) between the curve and two radii vectores. The area of the loop of the curve $y^2 = x^2\,\dfrac{a-x}{a+x}$ is $a^2(2-\frac{1}{2}\pi)$; and the whole area of the curve $r^2 = a^2\cos 2\theta$ is a^2.

6. Find the differential coefficients of the volume, and of the area of the surface generated by the revolution of a given curve about the axis of x. If the curve be that in (3), the volume generated by the loop is $\frac{3}{4}\pi a^3$, and the area of the surface generated is $3\pi a^2$.

7. The origin of polar coordinates is S, SA is (fig. 28) a fixed radius vector, P any point of a curve, $SP = r$, $ASP = \theta$, and U the volume generated by the revolution of ASP about SA; prove that $\dfrac{dU}{d\theta} = \frac{2}{3}\pi r^3 \sin\theta$.

The volume of the spherical sector included between a sphere and a cone of vertical angle 2α, with its vertex on the circumference, and its axis passing through the centre $= \dfrac{4\pi}{3} a^3 (1 + \cos^2\theta)\sin^2\theta$.

8. If NP (fig. 29) be the ordinate, PG the normal, terminated by the axis of x at any point P of a curve, and

NP be produced to p so that $Np = PG$, then the area of the surface generated by the revolution of the curve about the axis of $x = 2\pi \times$ the corresponding area of the curve traced out by p.

If the curve be the catenary $y = \frac{c}{2}(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}})$, the point p will lie on a catenary $y = \frac{c}{2} + \frac{c}{4}(\varepsilon^{\frac{2x}{c}} + \varepsilon^{-\frac{2x}{c}})$.

Contact of Curves.

If (x, y) be a point P (fig. 30) common to two curves, and if when we increase x to $x + \delta x$ the corresponding values of y in the two curves are $y + \delta y$, $y + \delta y'$, then the equations of the two being $y = f(x)$, $y = \phi(x)$, we have

$$y + \delta y = f(x) + f'(x)\,\delta x + f''(x)\frac{\delta x^2}{\lfloor 2} + \ldots + f^n(x + \theta\delta x)\frac{\delta x^n}{\lfloor n},$$

$$y + \delta y' = \phi(x) + \phi'(x)\,\delta x + \ldots + \phi^n(x + \theta\delta x)\frac{\overline{\delta x}|^n}{\lfloor n}.$$

Now $f(x) = \phi(x)$ since the point is common to both curves; also they will touch each other at the point if $f'(x) = \phi'(x)$, in which case they are said to have contact of the first order. If also $f''(x) = \phi''(x) \ldots f^n(x) = \phi^n(x)$, the two curves are said to have a contact of the n^{th} order. In such a case, if $OM = x$, $MP = y$, $ON = x + \delta x$, $NQ = y + \delta y$, $NQ' = y + \delta y'$, then

$$QQ' = \delta y' - \delta y = \{\phi^{n+1}(x + \theta\delta x) - f^{n+1}(x + \theta\delta x)\}\frac{(\delta x)^{n+1}}{\lfloor n+1};$$

i.e. if we call MN a small quantity of the first order of smallness, QQ' is a small quantity of the $(n+1)^{\text{th}}$ order; *i.e.* $\frac{QQ'}{MN^{n+1}}$ tends to a finite limit when δx is made $= 0$. If NQQ' meet the tangent at P in R, then

$$NR = f(x) + \delta x . f'(x);$$

therefore

$$RQ = f''(x+\theta\delta x)\frac{\delta x^2}{\lfloor 2}, \quad RQ' = \phi''(x+\theta\delta x)\frac{\delta x^2}{\lfloor 2};$$

therefore $$\frac{RQ}{RQ'} = \frac{f''(x+\theta\delta x)}{\phi''(x+\theta\delta x)},$$

or the limiting ratio of $RQ : RQ'$ is $f''(x) : \phi''(x)$.

Hence if two curves touch each other at (x, y); and, therefore, $f'(x) = \phi'(x)$, then if also $f''(x) = \phi''(x)$, the curves will deflect from the tangent at the same rate, or will have the same *curvature* at the point. This is, therefore, the same thing as having contact of the second order. This of course is obvious from the expression already found for the curvature $\frac{d^2y}{dx^2} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$; which is the same for all curves in which, at a given point (x, y), $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have the same values for the different curves. The circle of curvature may then be defined as the limiting position, when Q moves up to P of a circle which touches the curve at P, and passes through a neighbouring point Q of the curve, or since touching at P is the limit of meeting the curve in two points which ultimately coincide, we may also define the circle of curvature as the limit of the circle which meets the curve in three points Q, P, Q', when Q, Q' both move up to P. In just the same way any curve which meets a given curve in $(n+1)$ points of which P is one, will in the limit, when all the points move up to coincidence with P, have contact of the n^{th} order at P, and the order of contact which it will be possible to give to a curve of any defined species will depend on the number of points which suffice to fix such a curve, being one less than that number. Three points completely determine a circle, so that circles can only be made to have contact of the second order. Four points determine a parabola, and we can, therefore, draw a parabola having contact of the third order; and similarly an ellipse or hyperbola having contact of the fourth; and an epitrochoid of the fifth order.

Hence, to determine directly the position and dimensions of the circle of curvature of a given curve at a given point, we have to find a circle (1) passing through the point (x, y), (2) having the same value of $\frac{dy}{dx}$, (3) the same value of $\frac{d^2y}{dx^2}$, as the given curve has at (x, y).

Let, then, (a, b) be the coordinates of the centre (fig. 31), and ρ the radius of this circle, then if (XY) be any point on this circle $(X-a)^2+(Y-b)^2=\rho^2$, also the values of $\frac{dY}{dX}$, $\frac{d^2Y}{dX^2}$ at the point (XY) are found from the equations

$$X-a+(Y-b)\frac{dY}{dX}=0, \quad 1+\left(\frac{dY}{dX}\right)^2+(Y-b)\frac{d^2Y}{dX^2}=0.$$

Hence, to determine the circle of curvature of the given curve at (xy), we shall have the three equations

$$(x-a)^2+(y-b)^2=\rho^2, \quad (x-a)+(y-b)\frac{dy}{dx}=0,$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}=0,$$

so that
$$\frac{y-b}{1+\left(\frac{dy}{dx}\right)^2}=\frac{x-a}{-\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\frac{-1}{\frac{d^2y}{dx^2}},$$

and therefore

$$=\frac{\sqrt{(\overline{x-a}^2+\overline{y-b}^2)}}{\sqrt{\left[\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\right]}}=\frac{\pm\rho}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}},$$

so that a, b, ρ are completely determined.

These equations are more symmetrical if we take s the arc of the curve as independent variable. We shall then have

$$(x-a)^2+(y-b)^2=\rho^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(x-a)\frac{dx}{ds}+(y-b)\frac{dy}{ds}=0 \quad\ldots\ldots\ldots\ldots (2),$$

$$1+(x-a)\frac{d^2x}{ds^2}+(y-b)\frac{d^2y}{ds^2}=0 \quad\ldots\ldots\ldots (3);$$

a, b, ρ are functions of s determined by these three equations; if we change the position of (x, y) on the curve, we shall of course alter their value. To investigate the law of these changes, differentiate equation (1), taking account of (2), and we have

$$(x-a)\frac{da}{ds}+(y-b)\frac{db}{ds}=-\rho\frac{d\rho}{ds} \text{ .... } (4),$$

and similarly from (2),

$$\frac{da}{ds}\frac{dx}{ds}+\frac{db}{ds}\frac{dy}{ds}=0 \text{ .... } (5).$$

Now the centre of curvature will, as (x, y) moves on the curve itself trace out another curve called the *evolute* of the former, and the tangent at (a, b) to this curve makes with the axis of x the angle $\tan^{-1}\left(\frac{db}{da}\right)$, which by (5) $=\tan^{-1}\left(-\frac{dx}{dy}\right)$ the same as the normal at (x, y). But (a, b) lies on the normal at (x, y) (either by equation (2), or because the circle whose centre is (a, b) touches the curve at (x, y)), hence the tangent to the evolute at (a, b) is the normal to the curve at (x, y). Again by (5) and (2), we have

$$\frac{x-a}{\frac{da}{ds}}=\frac{y-b}{\frac{db}{ds}}=\frac{\sqrt{\{(x-a)^2+\overline{y-b}|^2\}}}{\sqrt{\left\{\left(\frac{da}{ds}\right)^2+\left(\frac{db}{ds}\right)^2\right\}}}=\frac{\pm\rho}{\frac{d\sigma}{ds}}$$

(σ being the arc of the evolute)

$$=\frac{(x-a)\frac{da}{ds}+(y-b)\frac{db}{ds}}{\left(\frac{da}{ds}\right)^2+\left(\frac{db}{ds}\right)^2}=\frac{-\rho\frac{d\rho}{ds}}{\left(\frac{d\sigma}{ds}\right)^2};$$

therefore $$\frac{d\rho}{ds}=\pm\frac{d\sigma}{ds};$$

therefore $$\rho=\pm\sigma+C.$$

This proves the property for which the locus of the

centres of curvature is called the evolute, namely, that the original curve may be generated by a point in an inextensible string which is kept tight and wrapped on, or unwrapped from the evolute.

For suppose OO' (fig. 32) is the curve locus of the centre of curvature, O the centre of curvature at P, O' at P', $PO=\rho$, $P'O'=\rho'$, σ the arc BO measured from some fixed point of the evolute, then since, in this figure, σ increases as ρ decreases, we must take the negative sign, and therefore

$$\rho+\sigma=C;$$

therefore $$\text{arc}\,BO+OP=\text{arc}\,BOO'+O'P,$$

or $$OP-O'P=\text{arc}\,OO',$$

or if PO be imagined a tight inextensible string, this string will just wrap on the evolute as P moves along the arc PP'. If, on the other hand, σ be measured in the opposite direction $=\text{arc}\,AO'O$, then σ and ρ increase together and the positive sign must be taken, hence

$$\rho=\sigma+C,$$

hence $$PO-\text{arc}\,AO=PO'-\text{arc}\,AO',$$

or $$PO-PO'=\text{arc}\,OO',$$

as before, of course. All this may easily be proved geometrically by considering a curve as the limit of a polygon, and considering the curve traced out by a point of a string unwrapped from the polygon.

The original curve is called an *involute* of the locus of the centres of curvature.

It is manifest that every curve will have a definite evolute, but an infinite number of *involutes* forming a system of *parallel curves*, such that P being any point of one of the curves, the others may be traced from it by always marking off the same fixed length along the normal at P.

Several other formulæ for the radius of curvature may be deduced from (1), (2), (3).

Since $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$, we have

$$\frac{dx}{ds}\,\frac{d^2x}{ds^2} + \frac{dy}{ds}\,\frac{d^2y}{ds^2} = 0\,;$$

therefore

$$\frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} = \frac{\dfrac{d^2x}{ds^2}}{-\dfrac{dy}{ds}} = \frac{\dfrac{dx}{ds}\,\dfrac{d^2y}{ds^2} - \dfrac{dy}{ds}\,\dfrac{d^2x}{ds^2}}{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 (=1)} = \frac{\sqrt{\left\{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2\right\}}}{\sqrt{\left\{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^3\right\}}}$$

$$= \frac{(y-b)\,\dfrac{d^2y}{ds^2} + (x-a)\,\dfrac{d^2x}{ds^2}}{(y-b)\,\dfrac{dx}{ds} - (x-a)\,\dfrac{dy}{ds}}\,.$$

But $$(x-a)\,\frac{dx}{ds} + (y-b)\,\frac{dy}{ds} = 0,$$

or

$$\frac{x-a}{\dfrac{dy}{ds}} = \frac{y-b}{-\dfrac{dx}{ds}} = \frac{(x-a)\,\dfrac{dy}{ds} - (y-b)\,\dfrac{dx}{ds}}{\sqrt{\left\{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2\right\}}} = \frac{\sqrt{(\overline{x-a}|^2 + \overline{y-b}|^2)}}{\sqrt{\left\{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2\right\}}},$$

therefore $$(y-b)\,\frac{dx}{ds} - (x-a)\,\frac{dy}{ds} = \pm\rho,$$

and since $$(y-b)\,\frac{d^2y}{ds^2} + (x-a)\,\frac{d^2x}{ds^2} = -1,$$

each of the quantities above $= \dfrac{\pm 1}{\rho}$; or

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} = \frac{\dfrac{d^2x}{ds^2}}{-\dfrac{dy}{ds}} = \frac{dx}{ds}\,\frac{d^2y}{ds^2} - \frac{dy}{ds}\,\frac{d^2x}{ds^2} = \sqrt{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}}.$$

These results are most readily found as follows:

$$\frac{dx}{ds} = \cos\psi,\quad \frac{dy}{ds} = \sin\psi\,;$$

therefore $$\frac{d^2x}{ds^2} = -\sin\psi\,\frac{d\psi}{ds} = -\frac{\sin\psi}{\rho},$$

if s, ψ increase together, so $\dfrac{d^2y}{ds^2} = \dfrac{\cos\psi}{\rho}$; therefore

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} = \frac{\dfrac{d^2x}{ds^2}}{-\dfrac{dy}{ds}} = \frac{dx}{ds}\frac{d^2y}{dx^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} = \sqrt{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}}.$$

So also
$$\frac{d^3x}{dx^3} = -\frac{\cos\psi}{\rho^2} + \frac{\sin\psi}{\rho^2}\frac{d\rho}{ds},$$
$$\frac{d^3y}{ds^3} = -\frac{\sin\psi}{\rho^2} - \frac{\cos\psi}{\rho^2}\frac{d\rho}{ds};$$

therefore
$$\left(\frac{d^3x}{ds^3}\right)^2 + \left(\frac{d^3y}{ds^3}\right)^2 = \frac{1}{\rho^4}\left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\},$$

so
$$\left(\frac{d^4x}{ds^4}\right)^2 + \left(\frac{d^4y}{ds^4}\right)^2 = \left(\frac{1}{\rho^3} - \frac{2}{\rho^3}\left.\overline{\frac{d\rho}{ds}}\right|^2 + \frac{1}{\rho^2}\frac{d^2\rho}{ds^2}\right)^2 + \left(\frac{3}{\rho^3}\frac{d\rho}{ds}\right)^2.$$

The centre of curvature may also be defined as the limiting position of the point of intersection of normals at consecutive points. The equation of the normal at (x, y) is

$$(Y - y)\frac{dy}{dx} + (X - x) = 0 \quad\text{.............(1)},$$

hence for the normal at a consecutive point $(x + \delta x,\ y + \delta y)$, the equation is

$$(Y - y - \delta y)\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\delta x + \ldots\right) + X - x - \delta x = 0 \ldots(2).$$

Hence at their point of intersection, making δx indefinitely small,

$$(Y - y)\frac{d^2y}{dx^2} - 1 - \left.\overline{\frac{dy}{dx}}\right|^2 = 0,$$

or
$$Y - y = \frac{1 + \left.\overline{\dfrac{dy}{dx}}\right|^2}{\dfrac{d^2y}{dx^2}},$$

and therefore
$$X - x = -\frac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}},$$

or (X, Y) is the same as (a, b). Hence also if O be this point (X, Y), P the point (x, y) at which the normal is drawn,

$$PO = \sqrt{(\overline{X-x}|^2 + \overline{Y-y}|^2)} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \div \frac{d^2y}{dx^2},$$

the radius of curvature as before.

Differential Calculus. VI.

1. What is meant by contact of the m^{th} order between two curves at a point. If two curves have contact of the n^{th} order at a point P, and NQQ' be an ordinate near P, meeting the curves in Q, Q', the limiting value of the ratio $QQ' : PQ^{n+1}$ when Q, Q' move up to P will be finite. If two curves have contact of an even order, they cross at the common point; if otherwise, not.

2. Explain why a circle cannot in general be made to have with a given curve at a given point a contact of higher order than the second; and prove that if it have at any point contact of the third order, the radius of curvature will be a maximum or minimum at the point. Prove that a parabola can be found having contact of the third order, and an ellipse or hyperbola having contact of the fourth.

3. Obtain the equations determining the centre and radius of the circle of curvature at any point (x, y) of a given curve. If s be the arc of the curve to (x, y), and ρ the radius of curvature, then will

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} = \frac{-\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}} = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}$$

$$= \sqrt{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}};$$

and $$\left(\frac{d^3x}{ds^3}\right)^2 + \left(\frac{d^3y}{ds^3}\right)^2 = \frac{1}{\rho^4}\left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\}.$$

4. If p be the perpendicular from the origin on the tangent, and ϕ the angle which the tangent makes with a fixed straight line, then the perpendicular on the normal will be $\frac{dp}{d\phi}$, and the radius of curvature $= p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}$. Hence, prove that the whole arc of a closed curve without singular points $= \int_0^{2\pi} p d\phi$.

5. The radius of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos\theta, b \sin\theta)$ is $(a^2 \sin^2\theta + b^2 \cos^2\theta)^{\frac{3}{2}} \div ab$, and the coordinates of the centre of curvature are $\frac{a^2 - b^2}{a} \cos^3\theta$, $\frac{b^2 - a^2}{b} \sin^3\theta$. The centre of curvature lies on the curve $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

6. If normals to a curve at P, Q meet in O, the limiting position of O when Q moves up to P will be the centre of curvature at P. Hence, prove that the radius of curvature of a parabola is $2a \sec^3\theta$, $4a$ being the latus rectum, and θ the angle which the normal makes with the axis.

7. In the curve $y = \frac{1}{2}c\left(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}}\right)$, the radius of curvature at a point where the tangent makes an angle ϕ with the axis of x is $c \sec^2\phi$, and if the normal at P meet the axis of x in C, PC will be equal and opposite to the radius of curvature at P.

8. The equation of the conic of closest contact at any point of a given curve referred to the tangent and normal at the point as coordinate axes is $Ax^2 + 2Hxy + By^2 = 2y$; the values of A, H, B being

$$\frac{1}{\rho}, \; -\frac{1}{3\rho}\frac{d\rho}{ds}, \; \text{and} \; \frac{1}{\rho} + \frac{2}{9\rho}\left(\frac{d\rho}{ds}\right)^2 - \frac{1}{3}\frac{d^2\rho}{ds^2},$$

where ρ is the radius of curvature at the point and s the arc to that point from some fixed point.

FUNDAMENTAL THEORY OF COUPLES.

By A. G. Greenhill, M.A.

Two equal, parallel, unlike forces make a couple.

The arm of a couple is the perpendicular distance between the forces.

The moment of a couple is the product of either force into the arm.

I. The moment of the forces of a couple about any axis perpendicular to the plane of the couple is constant and equal to the moment of the couple.

The moment of the forces (fig. 33) in the direction of the rotation of the couple about

$$O_1 \text{ is } P.O_1B_1 - P.O_1A_1 = P.A_1B_1,$$
$$O_2 \text{ is } P.O_2A_2 + P.O_2B_2 = P.A_2B_2,$$
$$O_3 \text{ is } P.O_3A_3 - P.O_3B_3 = P.A_3B_3.$$

II. A couple may be replaced by any other like couple of equal moment in the same or a parallel plane without altering the effect.

This is proved by shewing that two unlike couples in the same or parallel planes will balance if their moments are equal.

Let each force of one couple be P and of the other Q (fig. 34). Let $ABCD$ be the parallelogram formed by the lines of action of the forces.

Draw AM perpendicular to CD and AN to BC; then because the moments are equal

$$P \times AM = Q \times AN;$$

therefore
$$\frac{P}{Q} = \frac{AN}{AM} = \frac{AB}{AD};$$

therefore the resultant of P and Q acts in AC. Similarly the resultant of P and Q at C acts in CA, so that the resultants balance each other.

Hence the two couples balance each other; and, therefore, two like couples of equal moment in the same plane are equivalent.

If the forces of the couples were parallel we must suppose the couples like, and take a third couple unlike and of equal moment, with its forces not parallel to the forces of the two couples.

Then this couple will balance each of the two couples, and therefore the two couples are equivalent.

Next suppose two unlike couples of equal moment in parallel planes.

We may always replace them by two couples of equal moment having equal and parallel arms and forces.

Let any plane be drawn cutting the forces in the points A, B, C, D (fig. 35).

AB is equal and parallel to CD, and therefore $ABCD$ is a parallelogram of which the diagonals bisect each other at their point of intersection O.

The resultant of P at A and P at C is a parallel force $2P$ at O; and of P at B and P at D is an equal parallel but opposite force $2P$ at O, hence the system is in equilibrium.

Therefore two like couples of equal moment in parallel planes are equivalent.

Hence the resultant of any number of couples in the same or parallel planes is a couple, in a parallel plane of moment equal to the algebraical sum of the moments of the couples.

III. To represent a couple by a straight line—

(i) In point of application. The position of a couple with respect to a point is arbitrary, hence the straight line may be drawn from any point.

(ii) In direction. The straight line must be drawn at right angles to the plane of the couple, and in the direction of translation of a right-handed screw which has the same direction of rotation as the couple.

(iii) In magnitude. The length of the straight line must be proportional to the moment of the couple.

This straight line is called the axis of the couple; the axis is therefore a straight line drawn from any point at right angles to the plane of the couple, of length proportional to the moment of the couple, and in the direction of translation of a right-handed screw which turns in the same direction as the couple.

IV. To find the resultant of two couples represented by their axes OG and OH (fig. 36) describe a parallelogram on OG and OH as adjacent sides, then the diagonal OK will represent the axis of the resultant couple.

For, let the planes of the couples intersect in the line AB, and take AB as the arm of each couple, and P and Q as the forces.

Find the resultants R of P and Q at A, and of P and Q at B; the two resultants are equal and opposite, and constitute the resultant couple.

OK is perpendicular to the plane of this couple, and if $OG = P.AB$, $OH = Q.AB$, then will $OK = R.AB$; and therefore OK is the axis of the resultant couple.

Hence, to find the resultant of any number of forces, find the resultant of their axes by the parallelogramic law.

The resultant, therefore, of a system of couples in the same plane, or in parallel planes, is a couple of which the moment is the algebraic sum of the moments of the separate couples.

This is evident, independently, if we resolve each couple into two forces acting at the ends of an arm common to all the couples, when the forces of the resultant couple will be the algebraic sums of the forces of the component couples.

THE END.

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